FLOW OF A SECOND ORDER/GRADE FLUID DUE TO NON-COAXIAL ROTATION OF A POROUS DISK AND THE FLUID AT INFINITY

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The flow of an incompressible second order/grade fluid due to non-coaxial rotation of a porous disk and the fluid at infinity with the common angular velocity is studied. It is shown that there are physically acceptable solutions for both suction and blowing cases, depending on the sign of the material modulus $\alpha_f$. It is observed that the elasticity of the fluid causes the boundary layer thickness to increase in the case of suction, whereas it causes the boundary layer thickness to decrease in the case of blowing.

Key words: disk, fluid rotating at infinity, non-coaxial rotation, second order/grade fluid.

1. Introduction

If two disks or a disk and a fluid at infinity rotate about a common axis, it is possible to have solutions that are not axially symmetric. Berker (1979) showed that solutions of this type may be possible for the flow of a Newtonian fluid between two disks rotating with the same angular velocity about a common axis. For the classical linearly viscous fluid, in the case of the coaxial rotation of a disk and the fluid at infinity at different speeds, Lai et al. (1985) studied the flow including non-axisymmetric solutions. When there is a rotation about non-coincident axes, the flows induced cannot be axially symmetric. We refer the reader to a review article by Rajagopal (1992) regarding a comprehensive discussion of flows caused by the rotation of a disk or of two disks.

Coirier (1972) was the first to study the flow of a Newtonian fluid induced by non-coaxial rotation of a disk and the fluid at infinity with both the same and different angular velocities. Erdoğan (1976a, 1977) investigated the

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flow due to non-coaxial rotation of a porous disk and the fluid at infinity with the same and slightly different angular velocities, respectively. Murthy and Ram (1978) studied the effect of heat transfer on the MHD flow when a porous disk and the fluid at infinity rotate eccentrically with the same angular velocity. Kasiviswanathan and Rao (1987) presented an exact solution for the unsteady flow caused by a porous disk oscillating in its own plane. Erdoğan (1997) investigated the initial condition which makes the unsteady flow two-dimensional. He studied the unsteady flow caused by the rotation about non-coaxial axes while the disk and the fluid at infinity are initially rotating about a common axis. Hayat et al. (1999) obtained an exact analytic solution for the unsteady flow induced by the oscillations of a porous disk in its own plane. They also discussed the unsteady flow due to the porous disk oscillating and the fluid at infinity rotating about an axis parallel to their first rotation axis. Erdoğan (2000) found a solution for the unsteady flow produced by a disk executing oscillations in its own plane.

When the fluid considered is of non-Newtonian character, very little efforts have so far been made to examine the flow induced by a disk and a fluid at infinity rotating about non-coincident axes. Erdoğan (1976b) considered the case of rotation with the nearly same angular velocity and found an approximate analytic solution for a second order fluid. Ersoy (2000) investigated the MHD flow of a conducting, incompressible Oldroyd-B fluid due to non-coaxial rotation of a porous, insulated disk and the fluid at infinity with the same angular velocity. Erdoğan’s work (1997) on the Newtonian fluid has been recently extended by Siddiqi et al. (2001) to the case of a second grade fluid, i.e. $\alpha_1 > 0$ and $\alpha_1 + \alpha_2 = 0$, taking into account the effect of porosity in the presence of a magnetic field as well. They have considered the small values of the elastic parameter only, and hence reduced the order of the governing equation.

In this paper, the flow of a second order/grade fluid induced by the non-coaxial rotation of a porous disk and the fluid at infinity with the same angular velocity is examined. It is found that the existence of solutions is in connection with the sign of the material modulus $\alpha_1$ for the case of suction or blowing at the disk.

2. Basic equations and solution

In a Cartesian coordinate system, let us consider a porous disk in the $xy$-plane rotating counterclockwise at a constant rate of $\Omega$ about the $z$-axis perpendicular to the disk. A second order/grade fluid is present in the upper half-space $z \geq 0$. The axis of rotation of the fluid at infinity which rotates at equal angular velocity with the disk is parallel to $Oz$ axis and passes through the point $O'(x = 0, y = \ell)$ (see Fig. 1).
The Cauchy stress $T$ in an incompressible and homogeneous second order/grade fluid is given by Rivlin and Ericksen (1955)

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_i^2$$  \hspace{1cm} (2.1)

where $\mu$ is the dynamic viscosity of the fluid, $\alpha_1$ and $\alpha_2$ material moduli which are usually referred to as the normal stress coefficients. In the above representation, $p$ is the pressure, $I$ the identity tensor, and kinematical tensors $A_1$ and $A_2$ are defined through

$$A_1 = L + L^T,$$  \hspace{1cm} $L = \nabla \mathbf{v}$ \hspace{0.5cm} ($L_{ij} = v_{j,i}$),  \hspace{1cm} (2.2a)

$$A_2 = \frac{D}{Dt}A_1 + L \cdot A_1 + A_1 \cdot L^T$$  \hspace{1cm} (2.2b)

where $\mathbf{v}$ is the velocity vector, $\nabla$ the gradient operator, $D/Dt$ the material time derivative, and semicolon stands for covariant differentiation. We notice that if $\alpha_1 = \alpha_2 = 0$ the model Eq.(2.1) reduces to the classical linearly viscous fluid model.

The thermodynamical principles impose some restrictions on $\alpha_1$ and $\alpha_2$ (Dunn and Fosdick, 1974). In particular, the Clasius-Duhem inequality implies that
\[ \mu \geq 0, \quad \alpha_1 + \alpha_2 = 0 \quad (2.3a) \]

and the requirement that the specific Helmholtz free energy be a minimum in equilibrium implies that

\[ \alpha_1 \geq 0. \quad (2.3b) \]

The fluids characterized by above restrictions are called the second grade fluids in the literature. On the other hand, the model Eq.(2.1) is called a second order fluid model if it is not required to be compatible with thermodynamics (Rajagopal, 1981; 1984a). The sign of the coefficient \( \alpha_1 \) has been a subject of much controversy (Rajagopal, 1984a). Dunn and Fosdick (1974) demonstrated that a second grade fluid exhibits acceptable stability characteristics. Later, Fosdick and Rajagopal (1979) showed that the fluid exhibited anomalous behaviour not to be expected of any fluid of rheological interest if \( \alpha_1 < 0 \) and \( \alpha_1 + \alpha_2 \neq 0 \). On the other hand, experimental results are in good agreement when \( \alpha_1 < 0 \) and \( \alpha_1 + \alpha_2 \neq 0 \) (Rajagopal, 1984a). It is very important to bear in mind that solutions to steady flow problems can be found when \( \alpha_1 < 0 \). However, all these flows are not stable (Dunn and Rajagopal, 1995). Several such solutions corresponding to the case \( \alpha_1 < 0 \) are presented in the recent book of Truesdell and Rajagopal (2000). As a result, in any event, the results established for the case \( \alpha_1 > 0 \) have more value than the solution \( \alpha_1 < 0 \). In this study, we consider both positive and negative values of \( \alpha_1 \). Moreover, we notice that the material modulus \( \alpha_2 \) has no effect on the velocity field.

In the case of steady flow, the equations of motion and the continuity equation for an incompressible fluid, in the absence of body forces, are

\[ \rho (\mathbf{v} \cdot \nabla \mathbf{v}) = \nabla \cdot \mathbf{T}, \quad \nabla \cdot \mathbf{v} = 0 \quad (2.4) \]

where \( \rho \) is the density.

The boundary conditions for the velocity field are taken to be

\[ u = -\Omega y, \quad v = \Omega x, \quad w = \text{constant at } z = 0, \quad (2.5a) \]

\[ u = -\Omega (y - \ell), \quad v = \Omega x, \quad w = \text{constant at } z \to \infty \quad (2.5b) \]

where \( u, v, w \) denote the \( x, y, z \) components of the velocity, respectively.
It is natural to seek velocity field, compatible with the continuity equation, of the form

\[ u = -\Omega y + f(z), \quad v = \Omega x + g(z), \quad w = \text{constant}. \]  

(2.6)

This velocity field is even valid for the rotation about a common axis in the case of non-axially symmetric flow. Similarly, this velocity field was used by Rajagopal (1984b) studying the flow, which lacks symmetry, of a Newtonian fluid between two porous disks rotating about a common axis. The boundary conditions for \( f(z) \) and \( g(z) \) from Eqs.(2.5a, b)-(2.6) are

\[ f(0) = 0, \quad g(0) = 0, \quad f(\infty) = \Omega \ell, \quad g(\infty) = 0. \]  

(2.7)

From Eqs.(2.1), (2.2a, b) and (2.6), one has

\[ T_{xx} = -p + \alpha_2 f'^2, \quad T_{yy} = -p + \alpha_2 g'^2, \]

\[ T_{zt} = -p + (\alpha_2 + 2\alpha_1) \left(f'^2 + g'^2\right), \quad T_{yz} = \alpha_2 f'g', \]

\[ T_{xz} = \mu f' + \alpha_1 (\Omega g' + wf^*), \quad T_{yz} = \mu g' + \alpha_1 (-\Omega f' + wg^*) \]  

(2.8)

where a prime denotes differentiation with respect to \( z \). Substituting Eqs.(2.6) and (2.8) into the equations of motion, we obtain

\[ \frac{\partial p}{\partial x} = \rho \Omega (\Omega x + g) - \rho wf' + \mu f^* + \alpha_1 (\Omega g^* + wf^*), \]  

(2.9a)

\[ \frac{\partial p}{\partial y} = \rho \Omega (-\Omega y + f) - \rho wg' + \mu g^* + \alpha_1 (-\Omega f^* + wg^*), \]  

(2.9b)

\[ \frac{\partial p}{\partial z} = 2(\alpha_2 + 2\alpha_1) \left(f'f^* + g'g^*\right). \]  

(2.9c)

Using the conditions of integrability in Eqs.(2.9a-c), we get

\[ \rho (\Omega g - wf') + \mu f^* + \alpha_1 (\Omega g^* + wf^*) = \text{constant}, \]  

(2.10a)

\[ -\rho (\Omega f + wg') + \mu g^* - \alpha_1 (\Omega f^* - wg^*) = \text{constant}. \]  

(2.10b)
Defining $F(z) = f + ig$, Eqs.(2.10a, b) reduce to the following equation

$$\alpha_1 wF'' + (\mu - i\alpha_1 \Omega)F' - \rho wF' - ip\Omega F = -ip\Omega^2 \ell$$

(2.11)

with the conditions

$$F(0) = 0, \quad F(\infty) = \Omega \ell.$$  

(2.12)

Furthermore, all derivatives of $F(z)$ go to zero as $z \to \infty$ because the fluid at infinity is free of shear stress.

Let us make the variables non-dimensional by the following substitutions

$$\Gamma = F/\Omega \ell, \quad \zeta = \sqrt{\Omega/(2\nu)}z, \quad \beta = \alpha_1 \Omega/\mu, \quad e = w/(2\sqrt{\Omega \nu}).$$

(2.13)

Here $\nu$ denotes the kinematic viscosity of the fluid, $\beta$ and $e$ represent the elastic parameter and the suction-blowing parameter, respectively. As seen from Eq.(2.13) the case of suction corresponds to $e < 0$ and the case of blowing to $e > 0$. The non-dimensional equation becomes

$$\sqrt{2\beta} e \Gamma'' + (1 - i\beta)\Gamma' - 2\sqrt{2} e \Gamma' - 2i\Gamma = -2i$$

(2.14)

with the conditions as follows

$$\Gamma(0) = 0, \quad \Gamma(\infty) = 1, \quad \Gamma'(\infty) = \Gamma''(\infty) = \Gamma'''(\infty) = \ldots = 0.$$  

(2.15)

It is noticed that Eq.(2.14) is one order higher than the Navier-Stokes equations due to the viscoelasticity of the fluid. It would thus appear that the additional boundary condition must be imposed to determine the solution completely. The issue of difficulties with regard to prescribing boundary conditions are discussed in detail by Rajagopal (1994). Since the flow under consideration takes place in unbounded domain, we are able to overcome this difficulty by using asymptotic conditions and boundedness of solutions, as in the study of Rajagopal and Gupta (1984). They examined the existence of solutions that is tied in with the sign of material modulus $\alpha_1$ for the flow of an incompressible fluid of second order/grade past an infinite porous plate subject to either suction or blowing at the plate. They found that if the material modulus $\alpha_1 > 0$ it is possible to exhibit an exact solution which is asymptotic in nature for both suction and blowing at the plate. However, in the case of $\alpha_1 < 0$, they found that such solutions cannot exist for the blowing case.
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Fig. 2. Profiles of $f/\Omega \ell$ and $g/\Omega \ell$ for $\beta = 0.1$ and $e = 0, -0.2, -0.5, -1, -5$.

Fig. 3. Profiles of $f/\Omega \ell$ and $g/\Omega \ell$ for $e = -0.5$ and $\beta = 0, 0.2, 0.5, 1$.

Fig. 4. Profiles of $f/\Omega \ell$ and $g/\Omega \ell$ for $\beta = -0.5$ and $e = 0, 0.2, 0.5, 0.8$.

Fig. 5. Profiles of $f/\Omega \ell$ and $g/\Omega \ell$ for $e = 0.2$ and $\beta = 0, -0.5, -1$.

The characteristic equation of Eq. (2.14) is in the form of a cubic equation and has three roots. In order to obtain physically acceptable solutions to Eq. (2.14) under the conditions (2.15), this characteristic equation must have only one complex root with negative real part. Otherwise, the conditions Eq. (2.15) will not suffice to get physically acceptable solutions. It is for this
reason that there exist above mentioned solutions for $\alpha_j > 0$ in the case of suction, on the other hand, for $\alpha_j < 0$ in the case of blowing. The variations of $f/\Omega l$ and $g/\Omega l$ for various values of parameters are plotted against $\zeta$ in Figs. 2-5.

Fig. 6. Space curves consisting of the points about which the fluid layers rotate as a rigid body with a constant angular velocity $\Omega$.

3. Discussion

When a porous disk and a fluid at infinity rotate eccentrically with the same angular velocity, there exists a single point in each plane $z = \text{constant}$ where the velocity vector has only the axial component and about which the fluid layer rotates as a rigid body with a constant angular velocity. The coordinates of this point having only a constant axial velocity are given by
\[ x = -g(z)/\Omega, \quad y = f(z)/\Omega, \quad 0 \leq z < \infty. \] (3.1)

By considering that fluid layers rotate as a rigid body about the points given by Eq.(3.1) with a constant angular velocity \( \Omega \), the velocity components in the planes parallel to the \( xy \)-plane are calculated as depending on the coordinates. The locus of these points in space is a curve connecting the rotation center of the disk to that of the fluid rotating at infinity. Figure 6 shows these space curves for various values of the parameters \( e \) and \( \beta \). It is clear from these figures that the space curves move away from the \( yz \)-plane with increasing the suction-blowing parameter \( e \), which corresponds to either increasing blowing or decreasing suction. It is also noticed that for the blowing case, the space curve for a non-Newtonian fluid is nearer to the \( yz \)-plane than that corresponding to a Newtonian fluid. But for the case of suction an opposite effect is observed. At the region near to the disk the projections of these curves on the \( xy \)-plane are always in the second quadrant.

The graphs \( f/\Omega \ell \) and \( -g/\Omega \ell \) plotted versus \( \zeta \) in Figs.2 to 5 are the projections of the above mentioned space curves on the \( yz \)-plane and the \( xz \)-plane, respectively. These graphs show that there is a boundary layer near the porous disk, which is affected by the parameters \( e \) and \( \beta \). From these figures, it is concluded that the boundary layer thickness becomes small with an increase in the suction velocity (in an absolute sense), as expected. The reverse is true for the blowing case. Also, it is interesting to note that the fluid elasticity results in an increase in the boundary layer thickness for the suction case, whereas it results in an opposite effect for the blowing case.

**Nomenclature**

\( A_n \) - Rivlin-Ericksen tensor of rank \( n \)

\( D/\!\!Dt \) - material time derivative

\( e \) - suction-blowing parameter

\( i = \sqrt{-1} \)

\( I \) - identity tensor

\( \ell \) - eccentricity distance

\( p \) - pressure

\( T \) - Cauchy stress tensor

\( u, v, w \) - velocity components in Cartesian coordinate system

\( \mathbf{v} \) - velocity vector

\( x, y, z \) - Cartesian coordinates

\( \alpha_1, \alpha_2 \) - material moduli

\( \beta \) - elastic parameter

\( \mu \) - dynamic viscosity
\( \nu \) – kinematic viscosity
\( \rho \) – density
\( \Omega \) – common angular velocity of the disk and the fluid at infinity
\( \zeta \) – non-dimensional vertical distance
\( \nabla \) – gradient operator

Acknowledgments

The authors would like to express their gratitude to the referees for their valuable comments and suggestions.

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Received: August 03, 2001
Revised: January 16, 2002
International Journal of APPLIED MECHANICS AND ENGINEERING

2002 Volume 7 Number 4 ISSN 1425-1655

International Journal of AME is abstracted/indexed in:
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