Hydromagnetic Axisymmetric Stagnation Point Flow of a Reiner-Rivlin Fluid towards a Moving Plate

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Abstract: This study is concerned with the axisymmetric stagnation point flow of a Reiner-Rivlin fluid towards an infinite plate moving parallel to itself with constant velocity in the presence of a transverse uniform magnetic field. The governing partial differential equations were converted into ordinary nonlinear differential equations using similarity transformations available in the literature. It was found that the cross-viscous and magnetic effects depend on two non-dimensional numbers $S$ and $M$ respectively. We employed the Matlab routine bvp4c to obtain numerical solutions of the problem under consideration. The effects of controlling parameters on the dimensionless velocity temperature, wall shear stress and heat transfer rate were analyzed and discussed through graphs and tables.

Keywords: Stagnation point, magneto hydrodynamics, Reiner-Rivlin fluid, similarity transformation, numerical solution.

INTRODUCTION

The flow near a stagnation point has been a fundamental topic in fluid dynamics since the early times of viscous flow theory and many researchers because of its relevant applications have studied it. Some examples of practical applications include the cooling of continuous metallic plate drawn through quiescent or moving fluid, the prediction of skin friction and heat transfer near stagnation regions of bodies in high speed flows, the boundary layer along material handling conveyers, the design of thrust bearings and radial diffusers, the aero dynamic extrusion of plastic sheet, transpiration cooling and thermal oil recovery. Two-dimensional, axisymmetric and three-dimensional flows near stagnation point can be computed by solving systems of ordinary differential equations skillfully chosen functions and independent variables. Two of the most well-known are the two-dimensional and
axisymmetric stagnation point flows first studied by Hiemenz\textsuperscript{1} and Homann\textsuperscript{2}, respectively. Howarth\textsuperscript{3} and Davey\textsuperscript{4} extended the results of two dimensional and axisymmetric stagnation point flows to three-dimensional cases. We refer the reader to the articles by Blyth and Pozrikidis\textsuperscript{5}, Wang\textsuperscript{6} and the book by Schlichting and Gersten\textsuperscript{7} regarding detailed analysis of various results on stagnation point flows. Magneto hydrodynamic (MHD) stagnation point flows are relevant to many engineering applications such as petroleum engineering, chemical engineering, mhd pumps, heat exchangers and metallurgy industry. Neuringer and McIlroy\textsuperscript{8,9} were the first to present the solution for the hydromagnetic stagnation point flow. Ariel\textsuperscript{10} obtained an approximate solution of the hydromagnetic Hiemenz flow. The problem of steady forced convection flow of an electrically conducting and Chamkha\textsuperscript{11} solved heat generating/absorbing fluid near a stagnation point numerically. The effect of an externally applied uniform magnetic field on the two or three-dimensional stagnation point flow was given by Attia\textsuperscript{12,13} in the presence of uniform suction or injection. Abbasbandy and Hayat\textsuperscript{14} developed the homotopy analysis solution for the problem considered by Chamkha\textsuperscript{11}. Recently, Borelli et al.\textsuperscript{15} have studied numerically the steady three-dimensional stagnation point flow of an electrically conducting fluid over a flat plate. Another situation commonly observed in industrial applications is the stagnation point flow towards a moving plate. Authors like Rott\textsuperscript{16} and Glauert\textsuperscript{17} analyzed the two-dimensional stagnation point flow against an oscillating plate in its own plane. Wang\textsuperscript{18} and Libby\textsuperscript{19} considered the three-dimensional stagnation point flow against an infinite moving plate with a constant velocity. Weidman and Mahalingam\textsuperscript{20} solved the problem of axisymmetric stagnation point flow impinging on a flat plate oscillating in its own plane with suction a blowing. Javed et al.\textsuperscript{21} investigated the development of two-dimensional or axisymmetric stagnation flow of an incompressible viscous fluid over a moving plate with partial slip. Ja’fari and Rahimi\textsuperscript{22} considered the unsteady viscous flow and heat transfer near an axisymmetric stagnation point of an infinite plate with time dependent axial movement and with uniform transpiration. Recently, Demir and Barış\textsuperscript{23} have studied the stagnation flow of an electrically conducting incompressible viscous fluid towards a moving vertical plate in the presence of a constant magnetic field.

All the above-mentioned studies deal with flows of Newtonian fluids. However, over the last 50 years scientists because of fundamental and practical significance in the industrial and engineering applications have studied many new fluids not obeying Newtonian laws. In particular, these applications include extrusion processes, glass-fiber and paper production, electronic chips, application of paints, food processing and movement of biological fluids. Drilling mud, toothpaste, greases, polymer melts, cement slurries, paints, blood, clay coatings are some examples of non-Newtonian fluids.

An enormous of papers related to stagnation point flow and heat transfer of non-Newtonian fluids have written by researchers, e.g. Srivastava\textsuperscript{24}, Maiti\textsuperscript{25} Rajeswari and Rathna\textsuperscript{26}, Beard and Walters\textsuperscript{27}, Sarpkaya and Rainey\textsuperscript{28}, Soundalgekar and Vighnesam\textsuperscript{29}, Garg and Rajagopal\textsuperscript{30}, Massoudi and Ramazan\textsuperscript{31}, Garg\textsuperscript{32}, Dorrepaal et al.\textsuperscript{33}, Labropulu et al.\textsuperscript{34}, Ariel\textsuperscript{15,37}, Mahapatra and Gupta\textsuperscript{38}, Barış and Dokuz\textsuperscript{39}, Attia\textsuperscript{40}, Hayat et al.\textsuperscript{41}, Nadeem et al.\textsuperscript{42}, Bhattacharya and Layek\textsuperscript{43}, Lok et al.\textsuperscript{44}, Rehman et al.\textsuperscript{45}, Madhu and Kishan\textsuperscript{46}, Naganthran et al.\textsuperscript{47}. In this paper, we extend the analysis of Wang’s problem\textsuperscript{18} to a Reiner-Rivlin fluid in the presence of a constant magnetic field. The governing non-linear partial differential equations of momentum and energy were transformed into non-linear ordinary differential equations by means of appropriate similarity functions. The resulting problem was solved using the Matlab routine bvp4c. The results were shown graphically as well as in tabular form. Numerical results were compared with existing results in the literature and the agreement found to be excellent.
FORMULATION OF THE PROBLEM

Non-Newtonian fluids such as blood, thick oils, paints, colloidal suspensions and high polymer solutions are highly viscous fluids. They exhibit certain types of phenomena, which cannot be described by the classical Navier-Stokes equations. For example, if the fluid is contained between two coaxial cylinders, which rotated with respect to each other, the fluid will rise up the inner cylinder. Weissenberg\textsuperscript{48} drawn attention to the fact that a number of fluids exhibit such an effect. There are numerous models of non-Newtonian fluids suggested in the literature. To get some insight into their flow behaviour, it is preferable to restrict to a model with a minimum number of parameters in the constitutive equations. We choose the model of Reiner-Rivlin fluid for our study as it is the least complicated model, which exhibits non-Newtonian fluid behavior. In addition, the model can provide some insight into predicting the qualitative behavior of viscoelastic fluids though it allows only inelastic response. The constitutive equation for the Reiner-Rivlin fluid given by\textsuperscript{49}

\[
T = -pI + 2\mu_d + 4\mu_c d^2
\]

where \( T \) is the Cauchy stress tensor, \( p \) is the pressure, \( I \) is the identity tensor, \( \mu \) is the coefficient of viscosity, \( \mu_c \) is the coefficient of cross-viscosity and the rate of deformation tensor \( d \) is defined by

\[
2d = \nabla v + (\nabla v)^T
\]

where \( v \) is the velocity vector and \( \nabla \) is the gradient operator. Note that many liquids, notably polymer solutions, develop pressures normal to the shear direction when subjected to a shearing stress. The transverse stresses are observed in such liquids, which exhibit elasticity of shape. The term “cross-viscosity” is used to indicate these stresses\textsuperscript{50}.

![Figure 1: Physical model and coordinate system](image)

The orthogonal axisymmetric stagnation point flow against an infinite flat plate at \( z = 0 \) moving with constant velocity \( U_0 \) in the \( x \) direction was illustrated in Figure 1. A Reiner-Rivlin fluid flowing in the direction of negative \( z \)-axis approaches a moving plane at \( z = 0 \), and divides into streams proceeding away from the stagnation point at the origin. An external uniform magnetic field \( B_0 \) is applied in the \( z \)-direction. In the mathematical formulation of the problem, we shall assume that (i) the flow is steady and laminar, (ii) the fluid is incompressible, (iii) all the physical properties of the...
Hydromagnetic...  
B. Alanbel Ersin and S. Barış

fluid remain constants, (iv) gravitational effects are neglected, (v) the heat flux vector is represented by the Fourier law of heat conduction, (vi) the effects of dissipation and radiant heating are negligible, and (vii) the induced magnetic field is negligible. Under the above stated assumptions, the basic equation of the problem are as follows:

Continuity equation:
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]  
(3)

Equations of motion:
\[
\rho \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} - \sigma B_0^2 u
\]  
(4)
\[
\rho \left( \frac{\partial v}{\partial x} + u \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} - \sigma B_0^2 v
\]  
(5)
\[
\rho \left( \frac{\partial w}{\partial x} + u \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z}
\]  
(6)

Energy equation:
\[
\rho c_p \left( \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)
\]  
(7)

where \( u, v, w \) are the velocity components in the \( x, y, \) and \( z \)-directions, respectively, \( \rho \) is the density, \( c_p \) is the specific heat at constant pressure, \( k \) is the thermal conductivity, \( \sigma \) is the electrical conductivity, \( T \) is the temperature, and \( T_{ij} \)'s are the Cauchy stress tensor components.

Following transformations are introduced in accordance with Wang:
\[
u = U_0 f(\eta) + ax h'(\eta)
\]
\[
v = ay h'(\eta)
\]
\[
w = -2\sqrt{av} h(\eta)
\]

where \( a \) is the strength of the stagnation flow, \( \nu \) is the kinematic viscosity, \( \eta \) is the dimensionless similarity variable given by
\[
\eta = z \left( \frac{a}{\nu} \right)^{1/2}
\]  
(9)

and the prime denotes the differentiation with respect to \( \eta \).

The boundary conditions for the velocity field are
\[
u = 0, \quad w = 0 \quad \text{at} \quad z = 0
\]  
(10)
\[
u \to U_\infty = ax, \quad v \to V_\infty = ay, \quad w \to W_\infty = -2az, \quad \text{as} \quad z \to \infty
\]  
(11)
The conditions at infinity correspond to the rotational inviscid flow. With the velocity components given by Eq.(8), the continuity equation is identically satisfied.

Substituting Eq.(8) into the equations of motion (4)-(6) for the Reiner-Rivlin fluid given in Eq.(1) and eliminating the pressure term, we get

\[
(1 - 2Sh')h'' + 2hh' - h'^2 + Sh'' + M^2(1 - h') = -1 \tag{12}
\]

\[
(1 - 2Sh')f'' + 2hf' - h'f - Sh'f' - M^2f = 0 \tag{13}
\]

The physical parameters appearing in Eqs.(12) and (13) are defined as follows

\[
S = \frac{\alpha \mu_s}{\mu}, \quad M = \left( \frac{\sigma B_0^2}{a \rho} \right)^{1/2} \tag{14}
\]

where \( S \) is the parameter that describes the non-Newtonian behavior, and \( M \) is the magnetic parameter characterizing the strength of the imposed magnetic field. The appropriate boundary conditions for \( h(\eta) \) and \( f(\eta) \) are obtained from Eqs.(8)-(11) as

\[
h(0) = 0, h'(0) = 0, \quad h'(\infty) = 1 \tag{15}
\]

\[
f(0) = 1, f(\infty) = 0 \tag{16}
\]

It should be pointed out that for a Newtonian fluid \( (S = 0) \) Eqs.(12) and (13) together with the associated boundary conditions are the same as those obtained by Wang \(^{18}\) in the absence of a magnetic field.

From an engineering point of view, it is interesting to determine the shear stress on the plate in the \( x \)-direction. From the constitutive equation of a Reiner-Rivlin fluid, we obtain the dimensionless wall shear stress \( \tau_w \) as

\[
\tau_w = \frac{1}{\mu U_0 \sqrt{\alpha / \nu}} \left| \frac{T_x}{x=0} \right| = -f'(0) - \frac{aX}{U_0} h''(0) \tag{17}
\]

Due to the difference in temperature between the wall and the ambient fluid, heat transfer takes place. The boundary conditions for the energy equation are that the temperature equals \( T_w \) at the surface of the plate, and at large distances from the plate, \( T \) tends to \( T_\infty \), where \( T_\infty \) is the temperature of the ambient fluid. To obtain a similarity solution of the energy equation for the problem under discussion, we introduce the following similarity transformation:

\[
\frac{T - T_\infty}{T_w - T_\infty} = \theta(\eta) \tag{18}
\]

Substituting Eqs. (8) and (18) into Eq.(7) leads to the ordinary differential equation

\[
\theta'' + 2Pr h \theta' = 0 \tag{19}
\]

where \( Pr \) is the Prandtl number given by \( Pr = \mu c_p / k \). The boundary conditions are expressed in terms of \( \theta \) as
\[ \theta(0) = 1, \theta(\infty) = 0 \]  

(20)

For the problem under consideration, it is important to find the rate of heat transfer from the surface to the fluid. The Fourier law of heat conduction gives the heat transfer rate per unit area on the surface as follows:

\[ q_w = -k \left. \frac{\partial T}{\partial z} \right|_{z=0} = -k \left( \frac{a}{V} \right)^{\frac{1}{2}} (T_w - T_\infty) \theta'(0) \]  

(21)

RESULTS AND DISCUSSION

The nonlinear ordinary differential equations (12)-(13) and (19) subject to the boundary conditions (15)-(16) and (20) were numerically solved using the Matlab solver boundary value problem (bvp4c) for different values of physical parameters. The effects of the governing parameters on dimensionless velocity components, temperature, wall shear stress, and wall temperature gradient were examined. The bvp4c function in Matlab software employs a collocation method, which produces a continuous solution on an appropriate mesh. Error control is based on the residual of continuous solution. For more information about bvp4c solver and its performance in solving boundary value problems, the reader is referred to the book by Shampine et al.51 For this study, the convergence criterion was chosen to be \( 10^{-6} \). The asymptotic boundary conditions in Eqs.(15)-(16) and (20) were approximated by using the value of 5 for \( \eta_{\max} \) as follows:

\[ \eta_{\max} = 5, \ h'(5) = 1, \ f(5) = \theta(5) = 0 \]  

(22)

This confirms that all the numerical solutions approached the asymptotic values correctly. To validate the numerical method used in the present work, we made a comparison between our results and Wang’s results18 for the case of \( \text{S} = 0 \) and \( \text{M} = 0 \). The present results agree very well with those available in the literature as seen in the paper by Demir and Barış52. Therefore, we have concluded that the present code can be used with great confidence to study the problem discussed in this paper.

**Figure 2:** Lateral velocity component in the x-direction for \( \text{S} = 0.15 \)

**Figure 3:** Lateral velocity component in the x-direction for \( \text{M} = 1 \)
The velocity profiles are exhibited in Figures 2 to 7. Figure 2 displays the influence of the magnetic field on the $x$-wise velocity component for two different values of $\alpha x / U_0$ with keeping the non-Newtonian parameter $S$ fixed at 0.15. It is apparent from this figure that the velocity component in the $x$-direction continuously reduced with increasing values of the magnetic parameter $M$. Physically this is because increasing the magnetic field strength has a tendency to create a drag-like Lorentz force. This force acts as a resistance to the motion of the fluid in the $x$-direction. However, the reverse effect was noticed for the $y$-wise and $z$-wise velocity profiles, as presented in Figures 4 and 6. Again, from Figures 2 and 4, we arrive at the conclusion that the tangential velocity components approach the free stream values more quickly as $M$ increases. It is also noted that the lateral motion of the plate causes the $y$-wise velocity profiles to be different from the $x$-wise velocity profiles. The effects of the non-Newtonian parameter $S$, related to cross-viscosity, on the velocity components were depicted in Figures 3, 5 and 7. We observe that the cross-viscosity affects the velocity components in qualitatively the same way as the magnetic field does.
To investigate the effects of magnetic field and cross-viscosity on the temperature distribution, we plotted the function $\theta(\eta)$ against $\eta$ in Figures 8 and 9 for two different values of the Prandtl number $Pr$. From Figure 8, we notice that increasing the magnetic field strength causes the fluid temperature to decrease. Figure 9 shows that the effect of the non-Newtonian parameter $S$ is insignificant on the temperature profiles. By analyzing Figures 8 and 9, we observe that temperature profiles are reduced as the Prandtl number increases.

This is in agreement with the physical fact that the thermal boundary layer thickness decreases with increasing the Prandtl number. It is also noted that the temperature distribution is independent of plate translation since Eq.(19) doesn’t include the function $f(\eta)$ resulting from the translation of the plate.

### Table 1: Values of wall shear stress

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>$-\tau_w$</th>
<th>$ax/U_o = 0.1$</th>
<th>$ax/U_o = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0.807537</td>
<td>0.815652</td>
<td>0.830111</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1.179494</td>
<td>1.184762</td>
<td>0.830111</td>
</tr>
<tr>
<td>0.15</td>
<td>2</td>
<td>1.937161</td>
<td>1.939925</td>
<td>0.830111</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.939925</td>
<td>1.944137</td>
<td>0.830111</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.282762</td>
<td>0.288671</td>
<td>0.299279</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.521365</td>
<td>0.524991</td>
<td>0.531507</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.983077</td>
<td>0.984822</td>
<td>0.987905</td>
</tr>
</tbody>
</table>

The values of the wall shear stress $\tau_w$ tabulated in Table 1 for different values of the non-dimensional parameters. We conclude from this table that the wall shear stress increases with the increase in the magnetic parameter. Physically this can be explained as follows: We observe from Figure 2 that when $M$ increases, the velocity gradient at the moving plate increases. This causes $\tau_w$ to increase, and hence the force necessary to move the plate in the $x$-direction is greater. The cross-viscosity affects the wall shear stress in the same way as the magnetic field does (when regarded qualitatively).
Again from this table, it is evident that $\tau_w$ is increased as $\alpha x/U_0$ decreases. This is because the gradient of the $x$-wise velocity component increases with decrease in $\alpha x/U_0$.

**Table 2**: Values of wall heat transfer

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>$-\theta'(0)$</th>
<th>Pr = 0.7</th>
<th>Pr = 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.665377</td>
<td>1.545780</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.690930</td>
<td>1.634222</td>
<td></td>
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<tr>
<td>2</td>
<td>0</td>
<td>0.733856</td>
<td>1.789369</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.674455</td>
<td>1.562658</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.699802</td>
<td>1.651989</td>
<td></td>
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<tr>
<td>0</td>
<td>1</td>
<td>0.733856</td>
<td>1.789369</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.711152</td>
<td>1.674501</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.742343</td>
<td>1.809378</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.752980</td>
<td>1.834409</td>
<td></td>
</tr>
</tbody>
</table>

Table 2 provides the numerical data for the wall heat transfer rate per unit area $-\theta'(0)$ for selected values of the dimensionless parameters. From Table 2, we note that with an increase in either the parameter $S$ or the parameter $M$, the heat loss per unit area from the plate increases. This change in heat transfer rate is more pronounced for a large Prandtl number. This behavior is a consequence of the fact that the thermal boundary layer thickness decreases with increasing values of $Pr$.

**CONCLUSION**

The problem of mhd axisymmetric stagnation point flow of a Reiner-Rivlin fluid towards a moving plate was analyzed by a similarity solution approach. We applied the popular Matlab routine bvp4c to obtain the numerical solutions of the system of nonlinear ordinary differential equations governing the problem under discussion. The obtained results demonstrate the reliability of the bvp4c function in Matlab and encourage the application of the algorithm to fluid dynamics problems involving systems of nonlinear differential equations. The graphical and tabular presentation of the results revealed the effects of the relevant parameters on the velocity components, temperature, wall shear stress and wall heat transfer. It is hoped that the results of the present study may be useful for understanding of various technological problems related to mhd stagnation point flows on moving plates.

**REFERENCES**

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