Flow of a binary mixture of incompressible Newtonian fluids in a rectangular channel

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Abstract

The problems concerning some simple steady and unsteady flows of a mixture composed of two incompressible Newtonian fluids in an infinitely long channel of rectangular cross-section are examined. By means of finite Fourier sine transforms, the exact solutions of the field equations are obtained for the following four problems: (i) steady Couette flow in a rectangular channel, (ii) unsteady Couette flow in a rectangular channel, (iii) steady Poiseuille flow in a rectangular channel, (iv) unsteady Poiseuille flow in a rectangular channel.

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1. Introduction

The origin of the modern formulation of continuum thermomechanical theories of mixtures goes back to papers written by Truesdell [1]. He presented a comprehensive treatment of the
thermomechanics of interacting continua which discussed the appropriate forms for the balance of mass, momentum, energy and also the possible structure for the second law of thermodynamics. This work gave impetus to many researches on the theory of interacting continua and a rigorous and firm mathematical foundation has been developed. We refer the reader to the works of Bowen [2], Atkin and Craine [3,4], Bedford and Drumheller [5], and Rajagopal and Tao [6] regarding the historical development of the theory and detailed analysis of various results on this subject.

Adkins [7] formulated constitutive equations for the stresses in each constituent, and for diffusive force. He also examined some steady-state flows of compressible mixtures of non-Newtonian fluids. The continuum theory of compressible mixtures of Newtonian fluids was first considered by Green and Naghdi [8]. Müller [9] also studied Newtonian fluids and presented a thermomechanical theory for mixtures of fluids in which there are no chemical reactions. The theory is restricted to mixtures in which there exists a single temperature at each point. Dunwoody and Müller [10] were the first to investigate a more general theory where each constituent has its own temperature field. Eringen and Ingram [11,12] studied mixtures of chemically reacting fluids. The constitutive equations for an incompressible mixture of Newtonian fluids were derived by Mills [13] using the theory of Green and Naghdi [8]. Craine [14] examined the flow induced by the steady oscillations of an infinite plate in a mixture of two incompressible Newtonian fluids. In his subsequent study [15], he considered the same problem for a binary mixture of incompressible Newtonian hemihedral fluids. Beevers and Craine [16] extended the list of known solutions for a mixture of two incompressible Newtonian fluids and discussed in more detail methods for evaluating the response functions. Later some exact solutions for the flow of a binary mixture of incompressible Newtonian fluids were presented by Göğüş [17–19]. A theory was developed for binary mixtures of non-simple fluids by Iesan [20]. In a recent paper, Barış [21] obtained the exact solutions in series form for some simple unsteady unidirectional flows of a binary mixture of incompressible Newtonian fluids.

Recently, there has been a remarkable interest in flows of fluid mixtures due to the occurrence of these flows in industrial processes, particularly in lubrication practice. A familiar example is an emulsion which is the dispersion of one fluid within another fluid. Typical emulsions are oil dispersed within water or water within oil. Such emulsions are of considerable practical interest because synthetic fluids are more toxic than mineral oils and are uneconomical to use in applications requiring large quantities of lubricant, for which examples are metal working, mining, cutting and hydraulic fluids. Several problems relating to the mechanics of oil and water emulsions have been considered within the context of the mixture theory by Al-Sharif et al. [22], Chamnprasart et al. [23], and Wang et al. [24]. Another example where the fluid mixtures play an important role is in multigrade oils. In order to enhance the lubrication properties of mineral oils, such as the viscosity index, polymeric type fluids are added to the base oil [25].

In the present paper a binary mixture, each constituent of which is a chemically inert incompressible Newtonian fluid, is considered. The balance laws and relevant constitutive equations are presented in Section 2. In the subsequent sections we obtain the exact solutions for some simple steady and unsteady flows of the binary mixture under consideration in an infinitely long channel of rectangular cross-section. The solutions obtained in this paper are important not only in its own right as solutions of particular flows, but also serve as accuracy checks for the approximate solutions such as numerical, asymptotical and empirical.
2. Basic theory

A brief summary of the basic balance laws and the appropriate constitutive theory for a binary mixture of incompressible Newtonian fluids will be given here, for more details the reader should consult Atkin and Craine [3,4]. We consider a mixture of two interacting constituents, each of which is regarded as a continuum; we refer to the $b$th constituent as the continuum $s^{(b)}$. Throughout this paper $\beta$ takes the values 1 and 2. We assume that each point $x$ within the mixture is occupied simultaneously by one particle from each $s^{(\beta)}$. If $\mathbf{v}^{(\beta)}$ denotes the velocity vector of the $b$th constituent, the material time derivative $D^{(b)}/Dt$ is defined by

$$\frac{D^{(b)}}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}^{(b)} \cdot \nabla,$$

(2.1)

where $\nabla$ is the gradient operator.

The mean velocity of the mixture $\mathbf{w}$ and the total mass density of the mixture $\rho$ are given respectively by

$$\rho \mathbf{w} = \rho_1 \mathbf{v}^{(1)} + \rho_2 \mathbf{v}^{(2)},$$

(2.2)

$$\rho = \rho_1 + \rho_2,$$

(2.3)

where $\rho^{(b)}$ is the density of $s^{(\beta)}$ at time $t$, measured per unit volume of mixture.

The basic equations for a binary mixture in which the constituents have a common temperature and do not interact chemically are the following ones:

Continuity equations:

$$D^{(1)}\rho_1 + \rho_1 (\nabla \cdot \mathbf{v}^{(1)}) = 0, \quad D^{(2)}\rho_2 + \rho_2 (\nabla \cdot \mathbf{v}^{(2)}) = 0,$$

(2.4)

Equations of motion:

$$\rho_1 \frac{D^{(1)}\mathbf{v}^{(1)}}{Dt} = \nabla \cdot \mathbf{\sigma}^{(1)} - \mathbf{f} + \rho_1 \mathbf{F}^{(1)}, \quad \rho_2 \frac{D^{(2)}\mathbf{v}^{(2)}}{Dt} = \nabla \cdot \mathbf{\sigma}^{(2)} + \mathbf{f} + \rho_2 \mathbf{F}^{(2)},$$

(2.5)

where $\mathbf{f}$, $\mathbf{\sigma}^{(\beta)}$ and $\mathbf{F}^{(\beta)}$ are in turn diffusive force vector, partial stress and body force per unit mass of the $\beta$th constituent. For viscous fluids, in general, the interaction term $\mathbf{f}$ depends on the relative velocity, the density gradients, the temperature gradients, and possibly other quantities. Such interactions play a very important role in the nature of the solutions ([23,26–28]). In this study we shall assume that the interaction force incorporates only the effect of drag.

Consideration of the balance of angular momentum for $s^{(\beta)}$ shows that $\mathbf{\sigma}^{(\beta)}$ need not be symmetric although the balance of angular momentum for the mixture results in the symmetry of $\mathbf{\sigma}$, the total stress in the mixture, defined by

$$\mathbf{\sigma} = \mathbf{\sigma}^{(1)} + \mathbf{\sigma}^{(2)}.$$

(2.6)

In this work we shall concern ourselves with a mixture of two incompressible Newtonian fluids. Let the density of $s^{(\beta)}$ in its reference configuration be $\rho_{\beta 0}$, which in view of the assumed incompressibility is constant. Introducing the quantity $\phi_\beta$, which is the volume fraction of the $\beta$th fluid, and assuming that the mixture does not contain voids, it follows that
\[ \rho_1 = \phi_1 \rho_{10}, \quad \rho_2 = (1 - \phi_1) \rho_{20} \]

and hence
\[ \frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1. \]  \hspace{1cm} (2.8)

Using (2.3) and (2.8), it can be easily shown that
\[ \rho_1 = \frac{\rho_{10}(\rho_{20} - \rho)}{\rho_{20} - \rho_{10}}, \quad \rho_2 = \frac{\rho_{20}(\rho - \rho_{10})}{\rho_{20} - \rho_{10}}. \]  \hspace{1cm} (2.9)

Substituting Eq. (2.9) into Eqs. (2.4) and eliminating \( \partial \rho / \partial t \) between the resulting equations, we get
\[ (\rho_{20} - \rho) \text{tr}[d^{(1)}] + (\rho - \rho_{10}) \text{tr}[d^{(2)}] - \xi \cdot a = 0, \]  \hspace{1cm} (2.10)

where
\[ 2d^{(j)} = (\nabla v^{(j)})^T + \nabla v^{(j)}, \quad \xi = \nabla \rho, \quad a = v^{(1)} - v^{(2)}. \]  \hspace{1cm} (2.11)

In above equations, the superscript \( T \) and \( \text{tr} \) denote transpose and trace of a second-order tensor field, respectively. It is important to bear in mind that \( \nabla v^{(j)} \) in Eq. (2.11) is the second-order tensor field and its \( ij \)th component is taken as \( v^{(j)}_{ij} \), where semicolon stands for covariant differentiation.

The derivation of the constitutive equations appropriate to our binary mixture of incompressible Newtonian fluids has been outlined in the work of Atkin and Craine [4]. If the mixture is considered to be a purely mechanical system, that is, thermal effects are ignored, the relevant equations are
\[ A_{\beta} = A_{\beta}(\rho), \quad A = A(\rho), \]  \hspace{1cm} (2.12)

\[ p_1 = (\rho - \rho_{20}) \left( \rho_1 \frac{dA_1}{d\rho} + \lambda_1 \right), \quad p_2 = (\rho - \rho_{10}) \left( \rho_2 \frac{dA_2}{d\rho} - \lambda_2 \right), \quad p = p_1 + p_2, \]  \hspace{1cm} (2.13)

\[ f = xa - \lambda \xi, \]  \hspace{1cm} (2.14)

\[ \sigma^{(1)} = (-p_1 + \lambda_4 \text{tr}[d^{(1)}] + \lambda_5 \text{tr}[d^{(2)}]) I + 2\mu_1 d^{(1)} + 2\mu_2 d^{(2)} + \lambda_3 \Gamma, \]  \hspace{1cm} (2.15)

\[ \sigma^{(2)} = (-p_2 + \lambda_4 \text{tr}[d^{(1)}] + \lambda_2 \text{tr}[d^{(2)}]) I + 2\mu_1 d^{(1)} + 2\mu_2 d^{(2)} - \lambda_5 \Gamma, \]  \hspace{1cm} (2.16)

where \( p \) is the total pressure, \( A_{\beta} \) denotes the partial Helmholtz free energy of the \( \beta \)th constituent, and the Helmholtz free energy of the mixture \( A \) (total free energy) is defined by
\[ \rho A = \sum_{\beta} \rho_{\beta} A_{\beta} \]  \hspace{1cm} (2.17)

and the coefficients \( a, \lambda_1, \ldots, \lambda_5, \mu_1, \ldots, \mu_4 \) are functions of \( \rho \) and satisfy the inequalities
\[ a \geq 0, \quad \lambda_5 \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0, \quad \lambda_1 + \frac{2}{3} \mu_1 \geq 0, \quad \lambda_2 + \frac{2}{3} \mu_2 \geq 0. \]
\[
(\mu_3 + \mu_4) \leq 4\mu_1\mu_2, \quad \left[\dot{\lambda}_3 + \dot{\lambda}_4 + \frac{2}{3}(\mu_3 + \mu_4)\right]^2 \leq 4\left(\dot{\lambda}_1 + \frac{2}{3}\mu_1\right)\left(\dot{\lambda}_2 + \frac{2}{3}\mu_2\right).
\] (2.18)

The quantity \(\dot{\lambda}\) is a Lagrange multiplier associated with the constraint (2.10) and the relative spin \(\Gamma\) is given by

\[
2\Gamma = \{\langle \nabla v^{(1)} \rangle^T - \nabla v^{(1)}\} - \{\langle \nabla v^{(2)} \rangle^T - \nabla v^{(2)}\}.
\] (2.19)

Finally, neglecting the body forces, we shall derive the equations governing the flow of a mixture of two incompressible Newtonian fluids. For this purpose, inserting \(f, \sigma^{(1)}, \) and \(\sigma^{(2)}\) from Eqs. (2.14)–(2.16) into Eqs. (2.5), with the help of Eqs. (2.11) and (2.19), one gets the following equations of motion:

\[
M_1 \nabla^2 v^{(1)} + M_2 \nabla^2 v^{(2)} + M_5 \nabla(\nabla \cdot v^{(1)}) + M_6 \nabla(\nabla \cdot v^{(2)}) + (\nabla \cdot v^{(1)}) \nabla \dot{\lambda}_1 + (\nabla \cdot v^{(2)}) \nabla \dot{\lambda}_3 + (\nabla v^{(1)})^T \cdot \nabla M_1 + (\nabla v^{(2)})^T \cdot \nabla M_2 + \nabla v^{(1)} \cdot \nabla M_9
+ \nabla v^{(2)} \cdot \nabla M_{10} - \alpha (v^{(1)} - v^{(2)}) = \rho_1 \frac{\nabla v^{(1)}}{\nabla t} + \nabla p_1 - \dot{\lambda}\nabla \rho,
\] (2.20)

\[
M_3 \nabla^2 v^{(1)} + M_4 \nabla^2 v^{(2)} + M_7 \nabla(\nabla \cdot v^{(1)}) + M_8 \nabla(\nabla \cdot v^{(2)}) + (\nabla \cdot v^{(1)}) \nabla \dot{\lambda}_4 + (\nabla v^{(2)}) \nabla M_3 + (\nabla v^{(2)})^T \cdot \nabla M_4 + \nabla v^{(1)} \cdot \nabla M_{11} + \nabla v^{(2)} \cdot \nabla M_{12}
+ \alpha (v^{(1)} - v^{(2)}) = \rho_2 \frac{\nabla v^{(2)}}{\nabla t} + \nabla p_2 + \dot{\lambda} \nabla \rho,
\] (2.21)

where

\[
M_1 = \mu_1 + \frac{\dot{\lambda}_5}{2}, \quad M_2 = \mu_3 - \frac{\dot{\lambda}_5}{2}, \quad M_3 = \mu_4 - \frac{\dot{\lambda}_5}{2}, \quad M_4 = \mu_2 + \frac{\dot{\lambda}_5}{2},
\]

\[
M_5 = \dot{\lambda}_1 + \frac{\dot{\lambda}_5}{2}, \quad M_6 = \dot{\lambda}_3 + \frac{\dot{\lambda}_5}{2}, \quad M_7 = \dot{\lambda}_4 + \frac{\dot{\lambda}_5}{2}, \quad M_8 = \dot{\lambda}_2 + \frac{\dot{\lambda}_5}{2},
\]

\[
M_9 = \mu_1 - \frac{\dot{\lambda}_5}{2}, \quad M_{10} = \mu_3 + \frac{\dot{\lambda}_5}{2}, \quad M_{11} = \mu_4 + \frac{\dot{\lambda}_5}{2}, \quad M_{12} = \mu_2 - \frac{\dot{\lambda}_5}{2}.
\] (2.22)

In the subsequent sections, we shall obtain the exact solutions of the above equations for some simple steady and unsteady flows of a mixture composed of two incompressible Newtonian fluids in an infinitely long channel of rectangular cross-section.

3. Steady Couette flow in a rectangular channel

First, we consider the fully developed stage of the flow of a binary mixture of incompressible Newtonian fluids in an infinitely long channel of rectangular cross-section, the walls of which are situated at \(x = 0, L, y = 0, H.\) Throughout this paper it is supposed that \(L \gg H.\) It is assumed
that the flow is entirely driven by the motion of the wall at \( y = 0 \) with steady velocity \( U \) in the \( z \)-direction, the pressures far upstream and far downstream being kept equal throughout the motion. Thus the pressure gradients in the \( z \)-direction are zero.

We seek solutions in which the velocity vector of the \( \beta \)th fluid and densities are assumed to have the form

\[
\mathbf{v}^{(\beta)} = \{0, 0, w_{\beta \text{ISC}}(x, y)\}, \quad \rho_1 = \rho_1(x, y), \quad \rho_2 = \rho_2(x, y),
\]

where \( w_{\beta \text{ISC}} \) is the axial velocity of the \( \beta \)th fluid. With this assumption, it is shown that the equations of continuity (2.4) can be satisfied identically. Substituting Eqs. (3.1) into the \( x \) and \( y \)-components of the equations of motion (2.20) and (2.21), we get

\[
\lambda \frac{\partial p}{\partial x} = \frac{\partial p_1}{\partial x}, \quad -\lambda \frac{\partial p}{\partial y} = \frac{\partial p_2}{\partial x},
\]

\[
\lambda \frac{\partial p}{\partial y} = \frac{\partial p_1}{\partial y}, \quad -\lambda \frac{\partial p}{\partial y} = \frac{\partial p_2}{\partial y},
\]

With the use of Eqs. (2.9), (2.13) and (2.17), elimination of \( \partial v/\partial x \) between Eqs. (3.2)\(_1\) and (3.2)\(_2\), and that of \( \partial v/\partial y \) between Eqs. (3.3)\(_1\) and (3.3)\(_2\) give, respectively.

\[
(\rho - \rho_{10})(\rho_{20} - \rho) \frac{\partial \rho}{\partial x} \frac{d^2 (\rho A)}{d\rho^2} = 0,
\]

\[
(\rho - \rho_{10})(\rho_{20} - \rho) \frac{\partial \rho}{\partial y} \frac{d^2 (\rho A)}{d\rho^2} = 0,
\]

and since, in general, \( \rho \neq \rho_{10}, \rho \neq \rho_{20} \) and \( d^2 (\rho A)/d\rho^2 \neq 0 \) we deduce that \( \rho = \rho_0 = \text{const} \). As a consequence, the constitutive coefficients from (2.14)–(2.16) are constants. In the light of these arguments, the \( z \)-components of the equations of motion (2.20) and (2.21) reduce to

\[
M_1 \left( \frac{\partial^2 w_{\text{ISC}}}{\partial x^2} + \frac{\partial^2 w_{\text{ISC}}}{\partial y^2} \right) + M_2 \left( \frac{\partial^2 w_{2 \text{SC}}}{\partial x^2} + \frac{\partial^2 w_{2 \text{SC}}}{\partial y^2} \right) - \alpha (w_{\text{ISC}} - w_{2 \text{SC}}) = 0,
\]

\[
M_3 \left( \frac{\partial^2 w_{\text{ISC}}}{\partial x^2} + \frac{\partial^2 w_{\text{ISC}}}{\partial y^2} \right) + M_4 \left( \frac{\partial^2 w_{2 \text{SC}}}{\partial x^2} + \frac{\partial^2 w_{2 \text{SC}}}{\partial y^2} \right) + \alpha (w_{\text{ISC}} - w_{2 \text{SC}}) = 0.
\]

It is convenient at this point to introduce dimensionless variables and material constants. If \( \bar{f} \) is used to denote the dimensionless form of a quantity \( f \), it follows that

\[
\bar{M}_i = \frac{M_i}{\mu}, \quad \bar{z} = \frac{xH^2}{\mu}, \quad \bar{w}_{\beta \text{ISC}} = \frac{w_{\beta \text{ISC}}}{U}, \quad \bar{x} = \frac{x}{H}, \quad \bar{y} = \frac{y}{H},
\]

where \( \mu \) is the viscosity coefficient of the mixture. The dimensionless governing equations are obtained from Eqs. (3.6) and (3.7) by replacing variables and material constants by those given in Eq. (3.8), so they are not rewritten here.

The boundary conditions for the velocity fields are

\[
\bar{w}_{\beta \text{ISC}}(0, \bar{y}) = 0, \quad \bar{w}_{\beta \text{ISC}}(\bar{\delta}, \bar{y}) = 0, \quad \bar{w}_{\beta \text{ISC}}(\bar{x}, 0) = 1, \quad \bar{w}_{\beta \text{ISC}}(\bar{x}, 1) = 0,
\]

(3.9)
where $\delta = L/H$. Throughout this paper, henceforth for convenience, unless stated otherwise, we shall drop the bars that appear over the dimensionless quantities.

Finite Fourier sine transform will be used to solve the above two simultaneous partial differential equations with the boundary conditions (3.9). The finite Fourier sine transform of a function $f(x)$ defined for $0 < x < a$ is (29)

$$F_{S} \{ f(x) \} = \tilde{f}(k) = \int_{0}^{a} f(x) \sin \left( \frac{k \pi x}{a} \right) dx \quad k = 1, 2, 3, \ldots$$

(3.10)

with inverse transform

$$F_{S}^{-1} \{ \tilde{f}(k) \} = f(x) = \frac{2}{a} \sum_{k=1}^{\infty} \tilde{f}(k) \sin \left( \frac{k \pi x}{a} \right).$$

(3.11)

Application of the finite Fourier sine transform to Eqs. (3.6) and (3.7) with respect to $x$, taking Eqs. (3.9)1,2 into account, gives

$$M_{1} \frac{d^{2} \tilde{w}_{1SC}}{dy^{2}} + M_{2} \frac{d^{2} \tilde{w}_{2SC}}{dy^{2}} = \beta_{1} \tilde{w}_{1SC} - \beta_{2} \tilde{w}_{2SC},$$

(3.12)

$$M_{3} \frac{d^{2} \tilde{w}_{1SC}}{dy^{2}} + M_{4} \frac{d^{2} \tilde{w}_{2SC}}{dy^{2}} = -\beta_{3} \tilde{w}_{1SC} + \beta_{4} \tilde{w}_{2SC},$$

(3.13)

where

$$\beta_{1} = \alpha + \frac{M_{1} k^{2} \pi^{2}}{\delta^{2}}, \quad \beta_{2} = \alpha - \frac{M_{2} k^{2} \pi^{2}}{\delta^{2}}, \quad \beta_{3} = \alpha - \frac{M_{3} k^{2} \pi^{2}}{\delta^{2}}, \quad \beta_{4} = \alpha + \frac{M_{4} k^{2} \pi^{2}}{\delta^{2}},$$

(3.14)

subject to the transform of Eqs. (3.9)3,4

$$\tilde{w}_{jSC}(k, 0) = \frac{\delta \{ 1 - (-1)^{k} \}}{k \pi}, \quad \tilde{w}_{jSC}(k, 1) = 0.$$  

(3.15)

Solving Eqs. (3.12) and (3.13) and using the boundary conditions (3.15), we find

$$\tilde{w}_{jSC}(k, y) = \frac{\{ 1 - (-1)^{k} \} \delta \sinh \{ k \pi (1 - y) / \delta \}}{k \pi \sinh \{ k \pi / \delta \}},$$

(3.16)

Inverting Eq. (3.16), we obtain the following solution for $w_{jSC}(x, y)$

$$w_{jSC}(x, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\{ 1 - (-1)^{k} \} \sinh \{ k \pi (1 - y) / \delta \} \sin \{ k \pi x / \delta \}}{k \sinh \{ k \pi / \delta \}}.$$ 

(3.17)

It is recorded that there is no relative motion between the mixture constituents, as in the Couette flow between two infinite parallel plates (see, e.g., [4]). Moreover, the velocity field is identically to that resulting from the Navier–Stokes theory.
4. Unsteady couette flow in a rectangular channel

Now we shall examine unsteady Couette flow of a mixture of two incompressible Newtonian fluids in an infinitely long channel of rectangular cross-section. The mixture and the walls of the channel are initially at rest. The wall at $y = 0$ is suddenly accelerated from rest and moves in its own plane with a constant velocity $U$, while the other walls are held stationary. It is assumed that the flow is caused by the motion of the wall at $y = 0$, the pressures far upstream and far downstream being kept equal throughout the motion. Thus the pressure gradients in the z-direction are zero.

It seems reasonable to assume that the velocity distribution and total density in Cartesian coordinates are of the form

$$v^{(j)} = \{0, 0, w_{jC}(x, y, t)\}, \quad \rho_1 = \rho_1(x, y, t), \quad \rho_2 = \rho_2(x, y, t). \quad (4.1)$$

Substitution of Eqs. (4.1) into Eqs. (2.4) gives

$$\frac{\partial \rho_1}{\partial t} = \frac{\partial \rho_2}{\partial t} = 0, \quad (4.2)$$

thus $\partial \rho/\partial t = 0$ and $\rho = \rho(x, y)$. As made in the preceding section, elimination of $\partial \lambda/\partial x$ between the $x$-components of the equations of motion (2.20) and (2.21), and that of $\partial \lambda/\partial y$ between the $y$-components of the equations of motion (2.20) and (2.21) give respectively Eqs. (3.4) and (3.5), which imply that $\rho$ is a constant. Since $\rho$ has been proved to be constant, all of the coefficients in Eqs. (2.20) and (2.21) are constants. As a result, the dimensionless governing equations are as follows:

$$M_1 \left( \frac{\partial^2 \bar{w}_{1C}}{\partial x^2} + \frac{\partial^2 \bar{w}_{1C}}{\partial y^2} \right) + M_2 \left( \frac{\partial^2 \bar{w}_{2C}}{\partial x^2} + \frac{\partial^2 \bar{w}_{2C}}{\partial y^2} \right) - \bar{x}(\bar{w}_{1C} - \bar{w}_{2C}) = \bar{\rho}_1 \frac{\partial \bar{w}_{1C}}{\partial \bar{t}}, \quad (4.3)$$

$$M_3 \left( \frac{\partial^2 \bar{w}_{1C}}{\partial x^2} + \frac{\partial^2 \bar{w}_{1C}}{\partial y^2} \right) + M_4 \left( \frac{\partial^2 \bar{w}_{2C}}{\partial x^2} + \frac{\partial^2 \bar{w}_{2C}}{\partial y^2} \right) + \bar{x}(\bar{w}_{1C} - \bar{w}_{2C}) = \bar{\rho}_2 \frac{\partial \bar{w}_{2C}}{\partial \bar{t}}, \quad (4.4)$$

The boundary and initial conditions are

$$\bar{w}_{jC}(0, \bar{y}, \bar{t}) = 0, \quad \bar{w}_{jC}(\bar{x}, \bar{y}, 0) = 0, \quad \bar{w}_{jC}(\bar{x}, 0, \bar{t}) = 1, \quad \bar{w}_{jC}(1, \bar{y}, \bar{t}) = 0, \quad (4.5)$$

$$\bar{w}_{jC}(\bar{x}, \bar{y}, 0) = 0, \quad (4.6)$$

where

$$\bar{M}_i = \frac{M_i}{\mu}, \quad \bar{\rho}_j = \frac{\rho_j}{\rho_0}, \quad \bar{x} = \frac{zH^2}{\mu}, \quad \bar{w}_{jC} = \frac{w_{jC}}{U}, \quad \bar{x} = \frac{x}{H}, \quad \bar{y} = \frac{y}{H}, \quad \bar{t} = \frac{\mu t}{\rho_0 H^2}. \quad (4.7)$$

We first have to transform the problem into a problem with homogeneous boundary conditions. This can be achieved by decomposing $w_{jC}(x, y, t)$ into the steady-state Couette velocity profile $w_{jSC}(x, y)$, which are expected to prevail at large times, and the transient component $f_{j}(x, y, t)$:

$$w_{jC}(x, y, t) = w_{jSC}(x, y) - f_{j}(x, y, t), \quad (4.8)$$
The transient components satisfy the following differential equations
\[
M_1 \left( \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} \right) + M_2 \left( \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} \right) - \alpha(f_1 - f_2) = \rho_1 \frac{\partial f_1}{\partial t},
\]
(4.9)
\[
M_3 \left( \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} \right) + M_4 \left( \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} \right) + \alpha(f_1 - f_2) = \rho_2 \frac{\partial f_2}{\partial t},
\]
(4.10)
that are consistent with the boundary and initial conditions
\[
f_\beta(0, y, t) = 0, \quad f_\beta(\delta, y, t) = 0, \quad f_\beta(x, 0, t) = 0, \quad f_\beta(x, 1, t) = 0,
\]
(4.11)
\[
f_\beta(x, y, 0) = w_{\gamma SC}(x, y).
\]
(4.12)
Double finite Fourier sine transform will be used to solve the simultaneous partial differential Eqs. (4.9) and (4.10) with the boundary and initial conditions (4.11) and (4.12). If \( f(x, y) \) is a function of two independent variables \( x \) and \( y \), defined in a region \( 0 \leq x \leq a, 0 \leq y \leq b \), its double finite Fourier sine transform is defined by [29]
\[
F_S\{f(x, y)\} = \tilde{f}(m, n) = \int_0^a \int_0^b f(x, y) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) dx \, dy.
\]
(4.13)
The inverse transform is given by the double series
\[
F_{S}^{-1}\{\tilde{f}(m, n)\} = f(x, y) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{f}(m, n) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right).
\]
(4.14)
Taking the double finite Fourier sine transform of the system Eqs. (4.9–4.12) results in
\[
\frac{d\tilde{f}_1}{dt} = -\beta_1 \tilde{f}_1 + \beta_2 \tilde{f}_2,
\]
(4.15)
\[
\frac{d\tilde{f}_2}{dt} = \beta_3 \tilde{f}_1 - \beta_4 \tilde{f}_2,
\]
(4.16)
\[
\tilde{f}_\beta(m, n, 0) = \left\{ 1 - \left( -1 \right)^m \right\} \delta n / \pi^2 \mu_{mn},
\]
(4.17)
where
\[
\beta_1 = \frac{\alpha + M_1 \pi^2 \mu_{mn}}{\rho_1}, \quad \beta_2 = \frac{\alpha - M_2 \pi^2 \mu_{mn}}{\rho_2}, \quad \beta_3 = \frac{\alpha - M_3 \pi^2 \mu_{mn}}{\rho_2}, \quad \beta_4 = \frac{\alpha + M_4 \pi^2 \mu_{mn}}{\rho_2},
\]
(4.18)
\[
\mu_{mn} = \left( \frac{m^2}{\delta^2} + n^2 \right).
\]
Taking the double finite Fourier sine transform of the system Eqs. (4.9–4.12) results in
\[
\tilde{f}_1(m, n, t) = \exp \left\{ -0.5(\beta_1 + \beta_4)t \right\}
\]
\[
\times \left\{ \tilde{f}_1(m, n, 0) \cosh(0.5t \sqrt{\varepsilon}) + \frac{2\beta_2 \tilde{f}_2(m, n, 0) + (\beta_4 - \beta_1) \tilde{f}_1(m, n, 0)}{\sqrt{\varepsilon}} \sinh(0.5t \sqrt{\varepsilon}) \right\},
\]
(4.19)
\( \tilde{f}_2(m, n, t) = \exp\{-0.5(\beta_1 + \beta_4)t\} \times \left\{ \tilde{f}_2(m, n, 0) \cosh(0.5t\sqrt{\varepsilon}) + \frac{2\beta_3 \tilde{f}_1(m, n, 0) + (\beta_1 - \beta_4)\tilde{f}_2(m, n, 0)}{\sqrt{\varepsilon}} \sinh(0.5t\sqrt{\varepsilon}) \right\}, \)

(4.20)

where

\[ \varepsilon = \beta_1^2 + 4\beta_2\beta_3 - 2\beta_1\beta_4 + \beta_4^2. \]

(4.21)

The inverse transform gives the formal solution for \( f_\beta(x, y, t) \) in the form

\[ f_\beta(x, y, t) = \frac{4}{\delta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{f}_\beta(m, n, t) \sin\left(\frac{m\pi x}{\delta}\right) \sin(n\pi y). \]

(4.22)

Substituting Eqs. (3.17) and (4.22) into Eq. (4.8), we obtain the following solution for \( w_{\rho C}(x, y, t) \)

\[ w_{\rho C}(x, y, t) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{k} \sinh(k\pi(1 - y)/\delta) \sin(k\pi x/\delta) \]

\[ - \frac{4}{\delta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{f}_\beta(m, n, t) \times \sin\left(\frac{m\pi x}{\delta}\right) \sin(n\pi y). \]

(4.23)

5. Steady Poiseuille flow in a rectangular channel

In this section we study the steady flow of the binary mixture under consideration in an infinitely long channel of rectangular cross-section. The flow is driven by externally imposed pressure gradient in the \( z \)-direction, namely \(-\partial p/\partial z = p_z\).

We shall seek a solution, compatible with the mass balance Eq. (2.4), of the form

\[ \psi^{(0)} = \{0, 0, w_{\text{ISP}}(x, y)\}, \quad \rho_1 = \rho_1(x, y), \quad \rho_2 = \rho_2(x, y). \]

(5.1)

As in Section 3, it is proved that the total density and the coefficients appearing in Eqs. (2.20) and (2.21) become constants. Consequently, the equations of motion reduce to

\[ \mathcal{M}_1 \left( \frac{\partial^2 \tilde{w}_{1\text{SP}}}{\partial x^2} + \frac{\partial^2 \tilde{w}_{1\text{SP}}}{\partial y^2} \right) + \mathcal{M}_2 \left( \frac{\partial^2 \tilde{w}_{2\text{SP}}}{\partial x^2} + \frac{\partial^2 \tilde{w}_{2\text{SP}}}{\partial y^2} \right) - \bar{\alpha}(\tilde{w}_{1\text{SP}} - \tilde{w}_{2\text{SP}}) = -\phi_1, \]

(5.2)

\[ \mathcal{M}_3 \left( \frac{\partial^2 \tilde{w}_{1\text{SP}}}{\partial x^2} + \frac{\partial^2 \tilde{w}_{1\text{SP}}}{\partial y^2} \right) + \mathcal{M}_4 \left( \frac{\partial^2 \tilde{w}_{2\text{SP}}}{\partial x^2} + \frac{\partial^2 \tilde{w}_{2\text{SP}}}{\partial y^2} \right) + \bar{\alpha}(\tilde{w}_{1\text{SP}} - \tilde{w}_{2\text{SP}}) = \phi_1 - 1, \]

(5.3)

where

\[ \mathcal{M}_i = \frac{M_i}{\mu}, \quad \bar{\alpha} = \frac{\alpha H^2}{\mu}, \quad \tilde{w}_{\text{ISP}} = \frac{w_{\text{ISP}}}{p_z H^2}, \quad \bar{x} = \frac{x}{H}, \quad \bar{y} = \frac{y}{H}. \]

(5.4)
The adherence boundary conditions of the problem are follows:
\[ \bar{w}_{jsp}(0, \bar{y}) = 0, \quad \bar{w}_{jsp}(\delta, \bar{y}) = 0, \quad \bar{w}_{jsp}(\bar{x}, 0) = 0, \quad \bar{w}_{jsp}(\bar{x}, 1) = 0. \]  
(5.5)

Eqs. (5.2) and (5.3) can be transformed into the homogeneous ones by setting
\[ w_{jsp}(x, y) = \delta \beta \left\{ \frac{\sinh(0.5\eta(1 - y)) \sinh(0.5\eta y)}{\cosh(0.5\eta)} \right\} - \frac{y(y - 1)}{2\eta_2} - g_\beta(x, y), \]  
(5.6)

where
\[ \delta_1 = -\frac{2\eta_3(M_2 + M_4)}{\eta_2^2 x}, \quad \delta_2 = \frac{2\eta_3(M_1 + M_3)}{\eta_2^2 x}, \quad \eta = \sqrt{\frac{\eta_2 x}{\eta_1}} \]  
(5.7)

and
\[ \eta_1 = M_1M_4 - M_2M_3, \quad \eta_2 = M_1 + M_2 + M_3 + M_4, \]
\[ \eta_3 = (M_1 + M_2)(1 - \phi_1) - (M_3 + M_4)\phi_1. \]  
(5.8)

Note that the first two terms in the right-hand side of Eq. (5.6) is just the Poiseuille flow profile between two infinite plates placed at \( y = 0 \) and \( H \) (see, e.g., [16]). Substituting Eq. (5.6) into Eqs. (5.2), (5.3) and (5.5), we get
\[ M_1 \left( \frac{\partial^2 \bar{g}_1}{\partial x^2} + \frac{\partial^2 \bar{g}_1}{\partial y^2} \right) + M_2 \left( \frac{\partial^2 \bar{g}_2}{\partial x^2} + \frac{\partial^2 \bar{g}_2}{\partial y^2} \right) - \alpha(\bar{g}_1 - \bar{g}_2) = 0, \]  
(5.9)
\[ M_3 \left( \frac{\partial^2 \bar{g}_1}{\partial x^2} + \frac{\partial^2 \bar{g}_1}{\partial y^2} \right) + M_4 \left( \frac{\partial^2 \bar{g}_2}{\partial x^2} + \frac{\partial^2 \bar{g}_2}{\partial y^2} \right) + \alpha(\bar{g}_1 - \bar{g}_2) = 0, \]  
(5.10)

subject to
\[ g_\beta(0, y) = g_\beta(\delta, y) = \delta \beta \left\{ \frac{\sinh(0.5\eta(1 - y)) \sinh(0.5\eta y)}{\cosh(0.5\eta)} \right\} - \frac{y(y - 1)}{2\eta_2}, \]  
(5.11)
\[ g_\beta(x, 0) = g_\beta(x, 1) = 0 \]  
(5.12)

Application of the finite Fourier sine transform to Eqs. (5.9) and (5.10) with respect to \( y \), taking Eqs. (5.12) into account, gives
\[ M_1 \frac{d^2 \bar{g}_1}{dx^2} + M_2 \frac{d^2 \bar{g}_2}{dx^2} - (\alpha + M_1n^2\pi^2)\bar{g}_1 + (\alpha - M_2n^2\pi^2)\bar{g}_2 = 0, \]  
(5.13)
\[ M_3 \frac{d^2 \bar{g}_1}{dx^2} + M_4 \frac{d^2 \bar{g}_2}{dx^2} + (\alpha - M_3n^2\pi^2)\bar{g}_1 - (\alpha + M_4n^2\pi^2)\bar{g}_2 = 0, \]  
(5.14)

subject to the transform of Eq. (5.11)
\[ \bar{g}_\beta(0, k) = \bar{g}_\beta(\delta, k) = (1 - (-1)^k) \left( \frac{\eta^2 \delta \beta}{2k\pi a_k} + \frac{1}{k^3\pi^3\eta_2^2} \right), \]  
(5.15)

where \( a_k = \eta^2 + k^2\pi^2 \). Eqs. (5.13) and (5.14) which satisfy boundary conditions (5.15) are solved by the following analytical expressions
\( \bar{g}_\beta(x, k) = (1 - (-1)^k) \left( \frac{\cosh \left\{ k \pi (x - \frac{\delta}{2}) \right\}}{k^3 \pi^2 \eta_2 \cosh \left\{ k \pi \frac{\delta}{2} \right\}} - \gamma_\beta \eta_3 \cosh \left\{ \sqrt{\alpha_k} (x - \frac{\delta}{2}) \right\} \right), \) (5.16)

where \( \gamma_1 = -M_2 - M_4 \) and \( \gamma_2 = M_1 + M_3 \).

Inverting Eqs. (5.16), we obtain the following solution for \( g_\beta(x, y) \)

\[
g_\beta(x, y) = 2 \sum_{k=1}^{\infty} \bar{g}_\beta(x, k) \sin \{k \pi y\}. \tag{5.17}
\]

On inserting \( g_\beta(x, y) \) from Eq. (5.17) into Eq. (5.6), we obtain the velocity of the \( \beta \)th fluid

\[
w_{\beta SP}(x, y) = \delta_\beta \left\{ \frac{\sinh(0.5 \eta \{1 - y\}) \sinh(0.5 \eta \eta y)}{\cosh(0.5 \eta)} \right\} - \frac{y(y - 1)}{2 \eta_2} - 2 \sum_{k=1}^{\infty} \bar{g}_\beta(x, k) \sin(k \pi y). \tag{5.18}
\]

6. Unsteady Poiseuille flow in a rectangular channel

Finally, we discuss the problem of unsteady flow of a binary mixture of incompressible Newtonian fluids in an infinitely long channel of rectangular cross-section. Suppose that the rectangular channel is filled with a stationary mixture of two fluids. At the instant \( t = 0 \) constant pressure gradient in the \( z \)-direction, namely \( -\partial p/\partial z = P_Z \), is imposed and the fluids begin to flow.

Let us assume there exist a solution in the form

\[
v^{(\beta)} = \{0, 0, w_{\beta P}(x, y, t)\}, \quad \rho_1 = \rho_1(x, y, t), \quad \rho_2 = \rho_2(x, y, t). \tag{6.1}
\]

As in Section 4, it is verified that the total density and all of the coefficients in Eqs. (2.20) and (2.21) become constants. As a result, the dimensionless governing equations are as follows:

\[
M_1 \left( \frac{\partial^2 \bar{w}_{1P}}{\partial x^2} + \frac{\partial^2 \bar{w}_{1P}}{\partial y^2} \right) + M_2 \left( \frac{\partial^2 \bar{w}_{2P}}{\partial x^2} + \frac{\partial^2 \bar{w}_{2P}}{\partial y^2} \right) - \bar{g}(\bar{w}_{1P} - \bar{w}_{2P}) = \bar{\rho}_1 \frac{\partial \bar{w}_{1P}}{\partial t} - \phi_1, \tag{6.2}
\]

\[
M_3 \left( \frac{\partial^2 \bar{w}_{1P}}{\partial x^2} + \frac{\partial^2 \bar{w}_{1P}}{\partial y^2} \right) + M_4 \left( \frac{\partial^2 \bar{w}_{2P}}{\partial x^2} + \frac{\partial^2 \bar{w}_{2P}}{\partial y^2} \right) + \bar{g}(\bar{w}_{1P} - \bar{w}_{2P}) = \bar{\rho}_2 \frac{\partial \bar{w}_{2P}}{\partial t} + \phi_1 - 1. \tag{6.3}
\]

The boundary and initial conditions are

\[
\bar{w}_{\beta P}(0, \bar{y}, \bar{t}) = 0, \quad \bar{w}_{\beta P}(\bar{\delta}, \bar{y}, \bar{t}) = 0, \quad \bar{w}_{\beta P}(\bar{x}, 0, \bar{t}) = 0, \quad \bar{w}_{\beta P}(\bar{x}, 1, \bar{t}) = 0, \tag{6.4}
\]

\[
\bar{w}_{\beta P}(\bar{x}, \bar{y}, 0) = 0, \tag{6.5}
\]

where

\[
M_i = \frac{M_i}{\mu}, \quad \bar{\rho}_\beta = \frac{\rho_\beta}{\rho_0}, \quad \bar{x} = \frac{x H^2}{\mu}, \quad \bar{w}_{\beta P} = \frac{w_{\beta P} \mu}{\rho_0 H^2}, \quad \bar{\rho} = \frac{\rho}{\rho_0 H^2}. \tag{6.6}
\]

Note that all of the boundary and initial conditions given in Eqs. (6.4) and (6.5) are homogeneous, yet there exists a non-trivial solution, since the partial differential equations (6.2) and (6.3) are non-homogeneous.
We shall attempt to find a solution of the form

\[ w_{bP}(x, y, t) = w_{jSP}(x, y) - h_b(x, y, t). \] (6.7)

The components \( h_b(x, y, t) \) must satisfy the differential equations and boundary conditions which are obtained from Eqs. (4.9)–(4.11) by writing \( h_b(x, y, t) \) in place of \( f_b(x, y, t) \), but with modified initial conditions which now are:

\[ h_b(x, y, 0) = w_{jSP}(x, y). \] (6.8)

The procedure for determining \( h_b(x, y, t) \) is the same as that used in Section 4, so it is not repeated here. As expected, the solution given in Eq. (4.22) for \( f_b(x, y, t) \) is also valid for \( h_b(x, y, t) \) provided \( \tilde{h}_b(m, n, 0) \) is replaced with \( \tilde{h}_b(m, n, 0) \) which is given by the following analytical expression

\[ \tilde{h}_b(m, n, 0) = \left(1 - (-1)^m\right)\left(1 - (-1)^n\right)\delta \times \left(\frac{1}{\pi^4 n^3 \eta_1^2} - \frac{m}{\pi^4 n^3 (m^2 + n^2 \delta^2)} - \frac{\delta \eta^2}{\eta_1 n_2 a_n (a_n \delta^2 + m^2 \pi^2)}\right), \] (6.9)

where \( a_n = \eta^2 + n^2 \pi^2 \).

We now obtain the solution for the velocity of the \( \beta \)th fluid by going back through the various substitutions:

\[ w_{bP}(x, y, t) = \delta \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{h}_b(m, n, t) \sin\left(\frac{m\pi x}{\delta}\right) \sin(n\pi y). \] (6.10)

7. Numerical results and discussion

In this paper some simple steady and unsteady flows of a binary mixture composed of chemically inert incompressible Newtonian fluids in an infinitely long channel of rectangular cross-section are studied theoretically. Exact solutions for the system of coupled partial differential equations governing the velocity fields are obtained by using the finite Fourier sine transforms, which are extremely useful in solving a variety of partial differential equations in Cartesian coordinates.

At the outset it is necessary to discuss the reliability of the solutions given in Eqs. (4.23) and (6.10). As expected, these series are rapidly convergent for large values of time but slowly convergent for small values of time. However, it is important to note that series solutions (4.23) and (6.10) can also be used for small values of time provided number of terms in the series expansions is enough to yield satisfactory accuracy.

In order to make predictions based on foregoing analysis, it is necessary to know all of the material functions in the constitutive equations. Although determination of these functions for a mix-
ture is much more difficult than that for a single continuum, a good amount of literature has grown up around the problem of determining these functions due to the fact that the flow of mixtures is of great technical importance. For example, employing results obtained from the kinetic theory of fluids, Sampaio and Williams [30] were able to derive formulae for $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ in terms of the viscosities of the unmixed fluids and the volume fractions in the case of $\lambda_5 = 0$. In this work we benefit from the formulae suggested by Sampaio and Williams with a view to assigning the reasonable values to $\bar{M}_1, \bar{M}_2, \bar{M}_3$ and $\bar{M}_4$. To achieve this for a mixture composed of water and oil with water volume fraction $\phi_1$, at the outset we assume that the densities of unmixed fluids and the volume fractions are known. With the aid of Eqs. (2.3) and (2.7), knowledge of these quantities enables $\rho_1$, $\rho_2$ and $\rho_0$ to be calculated ([16]). Later, the material coefficients $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ can be determined using the viscosities of the unmixed fluids and the volume fractions ([30]). For the purpose of simulations, the following values are given to the dimensionless parameters:

$$\bar{M}_1 = 0.4868, \quad \bar{M}_2 = \bar{M}_3 = 0.2497, \quad \bar{M}_4 = 0.5132,$$
$$\bar{\rho}_1 = 0.8108, \quad \bar{\rho}_2 = 0.1892, \quad \phi_1 = 0.75. \quad (7.1)$$

In Figs. 1–6, for comparison, the velocity distributions for pure Newtonian fluid and constituents of binary mixture are plotted as a function of position for various values of $\bar{z}$ and $\bar{t}$, keeping the remaining parameters fixed at the values given in Eq. (7.1). For the sake of completeness, we also present the velocity distributions corresponding to pure Newtonian fluid. If one sets $\bar{M}_1 = \bar{M}_2 = \bar{M}_3 = \bar{M}_4 = 1/4$ and $\bar{\rho}_1 = \bar{\rho}_2 = \phi_1 = 1/2$ in Eqs. (4.23) and (6.10) these are obtained as follows:

**Unsteady Couette flow in a rectangular channel:**

![Fig. 1. Velocity profiles of Couette flow in a rectangular channel for $\delta = 2, \bar{x} = 1, \bar{z} = 10, \bar{t} = 0.05$ (--- pure Newtonian fluid, --- fluid 1, --- fluid 2).](image)
Unsteady Poiseuille flow in a rectangular channel:

\[ w_C(x, y, t) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{k} \frac{\sinh\{k\pi(1-y)/\delta\} \sin\{k\pi x/\delta\}}{k \sinh\{k\pi/\delta\}} \]

\[ -\frac{4\delta^2}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1 - (-1)^m)n}{m(m^2 + n^2\delta^2)} \exp(-\pi^2 w_{mn} t) \sin\left(\frac{m\pi x}{\delta}\right) \sin(n\pi y) \]  

(7.2)

Fig. 2. Velocity profiles of Couette flow in a rectangular channel for \( \delta = 2, \bar{x} = 1, \bar{z} = 10, \bar{t} = 0.18 \) (— pure Newtonian fluid, --- fluid 1, \( \cdots \) fluid 2).

Fig. 3. Velocity profiles of Couette flow in a rectangular channel for \( \delta = 2, \bar{x} = 1, \bar{z} = 50, \bar{t} = 0.05 \) (— pure Newtonian fluid, --- fluid 1, \( \cdots \) fluid 2).
\[ w_p(x, y, t) = \frac{y(1-y)}{2} - \frac{2}{\pi^3} \sum_{k=1}^{\infty} \left\{ \frac{1 - (-1)^k}{k^3} \right\} \cosh \left\{ k\pi \left( x - \frac{\delta}{2} \right) \right\} \sin \left\{ k\pi y \right\} - \frac{4}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1 - (-1)^m}{n^3} \frac{1 - (-1)^n}{m^3} \left( \frac{1}{m} - \frac{m}{m^2 + n^2\delta^2} \right) \times \exp(-\pi^2 w_{nw} t) \sin \left( \frac{m\pi x}{\delta} \right) \sin(n\pi y). \]
From Figs. 1–6, we observe how the velocity profiles grow with increasing time and approach asymptotically the steady-state velocity profiles. It is important to bear in mind that the assumptions of Eqs. (4.8) and (6.7) are convenient, since the unsteady problems investigated here approach the steady solutions as $t \to \infty$. On comparing Fig. 1 with Fig. 3 or Fig. 4 with Fig. 6, we arrive at the conclusion that with an increase in the coefficient of interaction $\bar{a}$, characterized by the drag force between the two constituents, the mixture tends to behave as a single continuum. As a result, it is not difficult to predict that the fluid particles belonging to both constituents have the same velocity at a given point in the mixture as $\bar{a} \to \infty$.

8. Conclusions

We have obtained some exact time-dependent solutions for the flows of a binary mixture including chemically inert incompressible Newtonian fluids in an infinitely long channel of rectangular cross-section, using the finite Fourier sine transforms. These exact analytical solutions in series form are rapidly convergent for large values of time but more slowly convergent for small values of time. If some conditions are satisfied, the series which is slowly convergent can also be used for small values of time without any difficulty. We note that for $\bar{M}_1 = \bar{M}_2 = \bar{M}_3 = \bar{M}_4 = 1/4$ and $\bar{\rho}_1 = \bar{\rho}_2 = \phi_1 = 1/2$, Eqs. (4.23) and (6.10) reduce to the classical solutions of a single incompressible Newtonian fluid. This provides a useful check.

References


