Analytical Solutions for Some Simple Flows of a Binary Mixture of Incompressible Newtonian Fluids

Serdar BARIS
Department of Mechanical Engineering, University of İstanbul
Avcilar, İstanbul-TURKEY
e-mail: bariss@itu.edu.tr

M. Salih DOKUZ
Department of Mechanical Engineering,
İstanbul Technical University,
Gümüşsuyu, İstanbul-TURKEY

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Abstract

The problems dealing with some simple flows of a mixture of two incompressible Newtonian fluids have been analysed. By using the theory of binary mixtures of Newtonian fluids, the equations governing the velocity fields are reduced to a system of coupled ordinary differential equations. In the case of non-inertial flow the analytical solutions of these equations have been obtained for the following three problems: (i) the parallel flow with a free surface; (ii) the flow between intersecting planes, one of which is moving; (iii) the flow between two coaxial moving cylinders.

Key words: Mixture, Newtonian fluid, Non-inertial flow.

Introduction

Recently, there has been remarkable interest in flows of fluid mixtures due to the occurrence of these flows in industrial processes, particularly in lubrication practice. A familiar example is an emulsion, which is the dispersion of one fluid within another fluid. Typical emulsions are oil dispersed within water or water within oil. Such emulsions are of considerable practical interest because synthetic fluids are more toxic than mineral oils and are uneconomical to use in applications requiring large quantities of lubricant, for example, metal working, mining, cutting and hydraulic fluids. Several problems relating to the mechanics of oil and water emulsions have been considered within the context of the mixture theory by Al-Sharif et al. (1993), Chamnprasart et al. (1993), and Wang et al. (1993). Another example where fluid mixtures play an important role is in multigrade oils. In order to enhance the lubrication properties of mineral oils, such as the viscosity index, polymeric type fluids are added to the base oil (Dai and Khonsari, 1994).

The origin of the modern formulation of continuum thermomechanical theories of mixtures goes back to papers written by Truesdell (1957). He presented a comprehensive treatment of the thermomechanics of interacting continua which discussed the appropriate forms for the balance of mass, momentum, energy and also the possible structure for the second law of thermodynamics. This work gave impetus to many studies on the theory of interacting continua and a rigorous and firm mathematical foundation has been developed. We refer the reader to the works of Bowen (1976), Atkin and Craine (1976b), Bedford and Drumheller (1983), and Rajagopal and Tao (1995) regarding the historical development of the theory and detailed analysis of various results on this subject.

In the present paper a binary mixture, each con-
stinent of which is an incompressible inert Newtonian fluid, is considered. In the following section the balance laws and relevant constitutive equations are briefly presented and then the equations governing the motion of the binary mixture are stated for the case of non-inertial flow. In the subsequent sections, we obtain the exact solutions for some simple flows of the binary mixture under consideration.

**Basic theory**

(i) **Kinematics and balance laws**

The governing equations are summarized in this section, for more details the reader should consult Craine (1971) and Atkin and Craine (1976a,b). Consider a mixture of two continua, in motion relative to each other. Let $X_\beta$ represent the position of a material point of the $\beta$ th constituent in its reference configuration. At any time $t$ each spatial point $x$ in the mixture is occupied simultaneously by one particle from each $\beta$. The motion of a binary mixture of components $\beta$ is denoted by

$$x = \phi_\beta(X_\beta, t), \quad t \geq 0, \quad \beta = 1, 2 \quad (1)$$

where the function $\phi_\beta$ is called the deformation function for the $\beta$ th constituent and is assumed to be sufficiently smooth so as to make the necessary mathematical operations correct. Throughout this paper the subscript $\beta$ takes the values 1, 2.

If $v_\beta$ denotes the velocity of the $\beta$ th constituent, the material derivative $D_\beta/Dt$ is defined by

$$\frac{D_\beta}{Dt} = \frac{\partial}{\partial t} + v_\beta \cdot \nabla \quad (2)$$

where $\nabla$ is the gradient operator.

Let the density of the $\beta$ th constituent, after mixing, be $\rho_\beta$, then the total density $\rho$ of the mixture is given by

$$\rho = \sum_\beta \rho_\beta \quad (3)$$

and the mean velocity, $w$, of the mixture is defined by

$$w = \frac{1}{\rho} \sum_\beta \rho_\beta v_\beta \quad . \quad (4)$$

The basic equations for a binary mixture in which the constituents have a common temperature $\mathcal{S}$ and do not interact chemically are the following:

**Continuity equations**

$$\frac{D_1 \rho_1}{Dt} + \rho_1 (\nabla \cdot v_1) = 0, \quad \frac{D_2 \rho_2}{Dt} + \rho_2 (\nabla \cdot v_2) = 0. \quad (5)$$

**Equations of motion**

$$\rho_1 \frac{D v_1}{Dt} = \nabla \cdot \sigma_1 - \mathbf{f} + \rho_1 \mathbf{F}_1, \quad \rho_2 \frac{D v_2}{Dt} = \nabla \cdot \sigma_2 + \mathbf{f} + \rho_2 \mathbf{F}_2. \quad (6)$$

**Energy equation**

$$\sum_\beta \rho_\beta \frac{D U_\beta}{Dt} = \rho r - \nabla \cdot \mathbf{q} + \mathbf{f} \cdot (v_1 - v_2) + \sum_\beta \text{tr} [\sigma_\beta \cdot (\nabla v_\beta)^T] \quad (7)$$

where the superscript $T$ and tr denote transpose and trace of a second-order tensor field, respectively. The quantities $U_\beta$, $\sigma_\beta$ and $\mathbf{F}_\beta$ are in turn internal energy per unit mass, partial stress and external body force acting on per unit mass of the $\beta$ th constituent. In addition, $r$, the heat supply per unit mass, and $\mathbf{q}$, the heat flux, refer to the mixture as a whole, and $\mathbf{f}$ denotes the diffusive force$^1$. It is important to bear in mind that the $ij$ th component of $\nabla v_\beta$ taken as $v_\beta^{j;i}$, where the semicolon stands for covariant differentiation.

Consideration of the balance of angular momentum for $R_1$ and $R_2$ shows that $\sigma_1$ and $\sigma_2$ need not be symmetric although the balance of angular momentum for the mixture results in the symmetry of $\sigma$, the total stress in the mixture, defined by

$$\sigma = \sigma_1 + \sigma_2. \quad (8)$$

Admissible thermomechanical processes in the mixture must be compatible with an entropy production inequality. If $S_1$ and $S_2$ are the entropies per unit mass of the constituents, then the Clausius-Duhem inequality may be written as follows (Green and Naghdi, 1969; Bowen and Wiese, 1969):

$^1$The diffusive force $\mathbf{f}$ may be interpreted as the drag exerted on one constituent due to the motion of the other (Craine, 1971).
\[
\sum_\beta \rho_\beta \frac{D_\beta S_\beta}{Dt} - \frac{\rho r}{3} + \nabla \cdot \left( \frac{q}{3} \right) \geq 0. \quad (9)
\]

(ii) Constitutive equations

In this work we shall concern ourselves with a mixture of two incompressible Newtonian fluids. Let the densities of \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), before mixing, be \( \rho_{10} \) and \( \rho_{20} \) respectively, which in view of the assumed incompressibility are constants. Introducing a composition factor \( \gamma \), defined as the proportion by volume of the constituent \( \mathcal{R}_1 \), and assuming that the mixture does not contain voids, it follows that

\[
p_1 = \gamma \rho_{10}, \quad p_2 = (1 - \gamma) \rho_{20} \tag{10}
\]

and hence

\[
\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1. \tag{11}
\]

By using (3) and (11), it can be easily shown that

\[
\rho_1 = \frac{\rho_{10} (\rho_{20} - \rho)}{\rho_{20} - \rho_{10}}, \quad \rho_2 = \frac{\rho_{20} (\rho - \rho_{10})}{\rho_{20} - \rho_{10}}. \tag{12}
\]

Substituting (12) into Eqs. (5) and eliminating \( \partial \rho / \partial t \) between them gives the relation

\[
(r_{20} - r) \text{tr}(\mathbf{d}_1) + (\rho - r_{10}) \text{tr}(\mathbf{d}_2) - \xi \cdot \mathbf{a} = 0 \tag{13}
\]

where

\[
2 \mathbf{d}_\beta = (\nabla v_\beta)^T + \nabla v_\beta, \quad \xi = \nabla \rho, \quad \mathbf{a} = \mathbf{v}_1 - \mathbf{v}_2. \tag{14}
\]

The derivation of the constitutive equations appropriate to our binary mixture of incompressible Newtonian fluids has been outlined in Atkin and Craine (1976a, b). If the mixture is considered to be a purely mechanical system, that is, thermal effects are ignored, the relevant equations are

\[
A_\beta = A_\beta(\rho), \quad A = A(\rho), \tag{15}
\]

\[
p_1 = (\rho - \rho_{10}) \left( \rho_1 \frac{dA_1}{d\rho} + \lambda \right), \tag{16}
\]

\[
p_2 = (\rho - \rho_{20}) \left( \rho_2 \frac{dA_2}{d\rho} - \lambda \right),
\]

\[
f = \alpha \mathbf{a} - \lambda \xi, \tag{17}
\]

\[
q = -k' \mathbf{a},
\]

\[
\sigma_1 = [-p_1 + \lambda_1 \text{tr}(\mathbf{d}_1) + \lambda_3 \text{tr}(\mathbf{d}_2)] \mathbf{I} + 2\mu_1 \mathbf{d}_1 + 2\mu_3 \mathbf{d}_2 + \lambda_5 \mathbf{\Gamma}, \tag{18}
\]

\[
\sigma_2 = [-p_2 + \lambda_4 \text{tr}(\mathbf{d}_1) + \lambda_2 \text{tr}(\mathbf{d}_2)] \mathbf{I} + 2\mu_4 \mathbf{d}_1 + 2\mu_2 \mathbf{d}_2 - \lambda_5 \mathbf{\Gamma} \tag{19}
\]

where \( A_\beta \) denotes the partial Helmholtz free energy, and the Helmholtz free energy \( A \) of the mixture (total free energy) is defined by

\[
A = \frac{1}{\rho} \sum_\beta \rho_\beta A_\beta \tag{20}
\]

and the coefficients \( \alpha, \lambda_1, \ldots, \lambda_5, k', \mu_1, \ldots, \mu_4 \) are functions of \( \rho \) and satisfy the inequalities

\[
\alpha \geq 0, \quad \lambda_1 \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0,
\]

\[
\lambda_1 + \frac{2}{3} \mu_1 \geq 0, \quad \lambda_2 + \frac{2}{3} \mu_2 \geq 0,
\]

\[
(\mu_3 + \mu_4)^2 \leq 4 \mu_1 \mu_2, \quad [\lambda_3 + \lambda_4 + \frac{2}{3} (\mu_3 + \mu_4)]^2
\]

\[
\leq 4 \left( \lambda_1 + \frac{2}{3} \mu_1 \right) \left( \lambda_2 + \frac{2}{3} \mu_2 \right). \tag{21}
\]

The quantity \( \lambda \) is a Lagrange multiplier associated with the constraint (13) and \( \mathbf{\Gamma} \) is given by

\[
2 \mathbf{\Gamma} = [(\nabla \mathbf{v}_1)^T - \nabla \mathbf{v}_1] - [(\nabla \mathbf{v}_2)^T - \nabla \mathbf{v}_2]. \tag{22}
\]

Finally, for the case of non-inertial flow \( (D_\beta v_\beta / Dt = 0) \), neglecting the body forces, we shall derive the equations governing the flow of a mixture of two Newtonian fluids. For this purpose, inserting \( \sigma_1, \sigma_2 \) and \( f \) from Eqs. (18), (19) and (17) into Eqs. (6), with the help of Eqs. (14) and (22), one gets the following equations of motion:

\[
2 \mathbf{\Gamma} = [(\nabla \mathbf{v}_1)^T - \nabla \mathbf{v}_1] - [(\nabla \mathbf{v}_2)^T - \nabla \mathbf{v}_2]. \tag{22}
\]

\[\text{See Beevers and Craine (1982).}\]
\[ M_1 \nabla^2 v_1 + M_5 \nabla (\nabla \cdot v_1) + (\nabla \cdot v_1) \nabla \lambda_1 + (\nabla v_1)^T \cdot (\nabla M_1) + (\nabla v_1) \cdot (\nabla M_9) \]

\[ + M_2 \nabla^2 v_2 + M_6 \nabla (\nabla \cdot v_2) + (\nabla \cdot v_2) \nabla \lambda_3 + (\nabla v_2)^T \cdot (\nabla M_2) + (\nabla v_2) \cdot (\nabla M_{10}) \]

\[ - \alpha (v_1 - v_2) = -\lambda \nabla \rho + \nabla p_1 , \quad (23) \]

\[ M_3 \nabla^2 v_1 + M_7 \nabla (\nabla \cdot v_1) + (\nabla \cdot v_1) \nabla \lambda_4 + (\nabla v_1)^T \cdot (\nabla M_3) + (\nabla v_1) \cdot (\nabla M_{11}) \]

\[ + M_4 \nabla^2 v_2 + M_8 \nabla (\nabla \cdot v_2) + (\nabla \cdot v_2) \nabla \lambda_5 + (\nabla v_2)^T \cdot (\nabla M_4) + (\nabla v_2) \cdot (\nabla M_{12}) \]

\[ + \alpha (v_1 - v_2) = \lambda \nabla \rho + \nabla p_2 \quad (24) \]

where

\[ M_1 = \mu_1 + \frac{\lambda_5}{2} , \quad M_2 = \mu_3 - \frac{\lambda_5}{2} , \quad M_3 = \mu_4 - \frac{\lambda_5}{2} , \quad M_4 = \mu_2 + \frac{\lambda_5}{2} , \]

\[ M_5 = \lambda_1 + \mu_1 - \frac{\lambda_5}{2} , \quad M_6 = \lambda_3 + \mu_3 + \frac{\lambda_5}{2} , \quad M_7 = \lambda_4 + \mu_4 + \frac{\lambda_5}{2} , \quad M_8 = \lambda_2 + \mu_2 - \frac{\lambda_5}{2} , \]

\[ M_9 = \mu_1 - \frac{\lambda_5}{2} , \quad M_{10} = \mu_3 + \frac{\lambda_5}{2} , \quad M_{11} = \mu_4 + \frac{\lambda_5}{2} , \quad M_{12} = \mu_2 - \frac{\lambda_5}{2} . \quad (25) \]

Note that, under isothermal conditions, the coefficients \( M_1 \) etc. appearing in (23) and (24) depend only on the total density \( \rho \), and hence spatial coordinates. In the subsequent sections, we shall obtain the exact solutions of the above equations for some simple flows of a binary mixture of incompressible Newtonian fluids.

**Parallel flow with a free surface**

First, we examine the flow of a film of a binary mixture of incompressible Newtonian fluids of uniform thickness \( \delta \). The ambient air is assumed to be stationary and, therefore, the flow is driven by externally imposed pressure gradients \( \partial p_1 / \partial x \) and \( \partial p_2 / \partial x \). Let the \( y \)-axis be directed normally to the plate, and the \( x \)-axis along this plate (see Figure 1).

![Figure 1. Basic geometry of the problem](image)

We shall seek a solution of the form

\[ v_{\beta x} = v_{\beta x}(y), \quad \rho = \rho(y) \quad (26) \]

where the function \( v_{\beta x} \) denotes the velocity component of the \( \beta \)th fluid in the \( x \) direction. With this
Thus, Eqs. (27) and (29) reduce to

\begin{equation}
M_1 v''_{1x} + M_2 v''_{2x} + M'_1 v'_1 x + M'_2 v'_2 x
- \alpha (v_{1x} - v_{2x}) = \frac{\varepsilon_1}{\partial x},
\end{equation}

\begin{equation}
\lambda \rho' = \frac{\partial p_1}{\partial y},
\end{equation}

\begin{equation}
M_3 v''_{1x} + M_4 v''_{2x} + M'_1 v'_1 x + M'_4 v'_4 x
+ \alpha (v_{1x} - v_{2x}) = \frac{\varepsilon_2}{\partial x},
\end{equation}

\begin{equation}
- \lambda \rho' = \frac{\partial p_2}{\partial y}.
\end{equation}

In the above equations, primes denote differentiation with respect to \( y \). With the use of Eqs. (12), (16) and (20), elimination of \( \partial \lambda / \partial y \) between Eqs. (27) and (29) gives

\begin{equation}
(p - p_{10}) (p - \rho_0) \frac{d \rho}{dy} \frac{d^2 (\rho A)}{dy} = 0
\end{equation}

and since, in general, \( \rho \neq \rho_{10}, \rho \neq \rho_0 \) and \( d^2 (\rho A)/dy^2 \neq 0 \) we deduce that \( \rho \) is a constant. As a result, the coefficients \( M_1 \) etc. in (27) and (29) are constants. It also follows that the quantities \( p_1 \) and \( p_2 \) are not functions of \( y \). Then, from Eqs. (27) and (29), it is evident that the pressure gradients are constants, i.e. \( \partial p_1 / \partial x = -p_{10} \) and \( \partial p_2 / \partial x = -p_{20} \). Thus, Eqs. (27) and (29) reduce to

\begin{equation}
M_1 v''_{1x} + M_2 v''_{2x} - \alpha (v_{1x} - v_{2x}) = p_{10},
\end{equation}

\begin{equation}
M_3 v''_{1x} + M_4 v''_{2x} + \alpha (v_{1x} - v_{2x}) = p_{20}.
\end{equation}

It is convenient at this point to introduce dimensionless variables and material constants. If \( \bar{f} \) is used to denote the dimensionless form of a quantity \( f \), it follows that

\begin{equation}
\bar{y} = \frac{y}{\bar{y}}, \quad \bar{M}_i = \frac{M_i}{\mu}, \quad \bar{v}_{\beta x} = \frac{v_{\beta x} \mu}{p \alpha},
\end{equation}

\begin{equation}
\bar{\alpha} = \frac{\alpha \mu^2}{\mu}, \quad \bar{Q} = \frac{Q \mu}{p \alpha},
\end{equation}

where \( \mu \) is the viscosity of the mixture and \( Q \) is the volume flux of the mixture per unit distance normal to the plane of flow. In addition, it is assumed that the pressure gradients imposed on mixture components are the same, i.e. \( p_{10} = p_{20} = p_0 \). Thus the dimensionless governing equations are as follows:

\begin{equation}
\bar{M}_1 \bar{v}''_{1x} + \bar{M}_2 \bar{v}''_{2x} - \bar{\alpha} (\bar{v}_{1x} - \bar{v}_{2x}) = -1,
\end{equation}

\begin{equation}
\bar{M}_3 \bar{v}''_{1x} + \bar{M}_4 \bar{v}''_{2x} + \bar{\alpha} (\bar{v}_{1x} - \bar{v}_{2x}) = -1.
\end{equation}

Throughout this paper, henceforth for convenience, unless stated otherwise, we shall drop the bars that appear over the dimensionless quantities.

Subtracting \( M_4 \) times Eq. (35) from \( M_2 \) times Eq. (36), and \( M_3 \) times Eq. (35) from \( M_1 \) times Eq. (36), we get the following equations, respectively

\begin{equation}
\eta_1 \bar{v}''_{1x} - \alpha (M_2 + M_4) (v_{1x} - v_{2x}) = M_2 - M_4,
\end{equation}

\begin{equation}
-\eta_1 v''_{2x} - \alpha (M_1 + M_4) (v_{1x} - v_{2x}) = M_1 - M_3.
\end{equation}

and the sum of above equations is

\begin{equation}
\eta_1 (v''_{1x} - v''_{2x}) - \alpha \eta_2 (v_{1x} - v_{2x}) = \eta_3
\end{equation}

where

\begin{equation}
\eta_1 = M_1 M_4 - M_2 M_3, \quad \eta_2 = M_1 + M_2 + M_3 + M_4,
\end{equation}

\begin{equation}
\eta_3 = M_1 + M_2 - M_3 - M_4.
\end{equation}

Hereafter, we shall assume that \( \alpha \neq 0, \eta_1 \neq 0 \) and \( \eta_2 \neq 0 \).

The boundary conditions for the velocity fields are

\begin{equation}
v_{\beta x}(0) = 0, \quad v_{1x}(1) - v_{2x}(1) = W
\end{equation}
where $W$ is a constant to be determined later. Equation (39), which satisfies boundary conditions (41), is solved by the following simple analytical expression

$$v_{1x} - v_{2x} = C_1 \cosh(\gamma_1 y) + C_2 \sinh(\gamma_1 y) - \frac{\eta_3}{\alpha \eta_2} y$$

(42)

where $\gamma_1 = \sqrt{\alpha \eta_2 / \eta_1}$, and the constants $C_1$ and $C_2$ are

$$C_1 = \frac{\eta_3}{\alpha \eta_2}, \quad C_2 = \frac{W}{\sinh(\delta \gamma_1)} - C_1 \tanh\left(\frac{\delta \gamma_1}{2}\right).$$

Substituting Eq. (42) into Eqs. (37)-(38) and solving them, we have, respectively

$$v_{1x} = \frac{(M_2 + M_4)}{\eta_2} [C_1 \cosh(\gamma_1 y) + C_2 \sinh(\gamma_1 y)] + C_3 y + C_4 - \frac{\nu^2}{\eta_2},$$

(44)

$$v_{2x} = -\frac{(M_1 + M_3)}{\eta_2} [C_1 \cosh(\gamma_1 y) + C_2 \sinh(\gamma_1 y)] + C_5 y + C_6 - \frac{\nu^2}{\eta_2}$$

(45)

where $C_3, \ldots, C_6$ are the constants of integration. Boundary conditions (41) are not sufficient for determining these constants in a unique way. It would thus appear that the additional boundary condition must be imposed. This is a free-surface condition, that is, the atmospheric shear stress, which is assumed to be negligible (no wind, negligible air viscosity), must be equal to the total shear stress, $\sigma_{xy}$, of the mixture at $y = 1$. Thus

$$\sigma_{xy}(1) = (M_1 + M_3)v_{1x}'(1) + (M_2 + M_4)v_{2x}'(1) = 0.$$  

(46)

From conditions (41) and (46), we find that

$$C_3 = \frac{1}{\eta_2}\left\{(M_2 + M_4)\left[C_1 + W - C_1 \cosh(\gamma_1) - C_2 \sinh(\gamma_1)\right] + 2\right\},$$

$$C_5 = \frac{1}{\eta_2}\left\{- (M_1 + M_3)\left[C_1 + W - C_1 \cosh(\gamma_1) - C_2 \sinh(\gamma_1)\right] + 2\right\},$$

(47)

The volume rate of flow per unit length in the $z$-direction is

$$Q = \int_0^1 v_{1x} dy + \int_0^1 v_{2x} dy.$$  

(48)

Inserting $v_{1x}$ and $v_{2x}$ from Eqs. (44) and (45), with the aid of Eq. (47), into Eq. (48) yields

$$W = \gamma_1 \left[C_1 + \frac{4 - 3 \eta_2}{3(M_1 - M_2 + M_3 - M_4)} \coth\left(\frac{\gamma_1}{2}\right) - 2C_1\right]$$  

(49)

It is obvious from Eq. (49) that the value of constant $W$ in Eqs. (43) and (47) can be determined by experimental measurement of $Q$.

**Flow between intersecting planes, one of which is moving**

In this section, we consider the slow motion of a mixture of two incompressible Newtonian fluids near a corner of plane rigid walls, one of which is stationary and the other moving. The flow is caused by the motion of the wall at $\theta = 0$. The moving wall is made of a porous material through which the fluids are injected with constant velocities $V_1$ and $V_2$ (see Figure 2).

It seems reasonable to assume that the velocity distribution and total density in planar polar coordinates $(r, \theta)$ are of the form

$$v_\beta = [v_{\beta r}(r, \theta), v_{\beta \theta}(r, \theta)], \quad \rho = \rho(r, \theta)$$  

(50)

![Figure 2. Sketch of flow geometry and coordinate system](image_url)
where \( v_{\beta r} \) and \( v_{\beta \theta} \) denote the velocity components of the \( \beta \)th fluid in the directions of \( r \) and \( \theta \), respectively. By defining the stream function \( \psi_{\beta}(r, \theta) \), such that

\[
\psi_{\beta r} = \frac{1}{r} \frac{\partial \psi_{\beta}}{\partial \theta}, \quad \psi_{\beta \theta} = -\frac{\partial \psi_{\beta}}{\partial r}
\]  

(51)

the equation \( \nabla \cdot \mathbf{v} = 0 \) is satisfied automatically. In this case, we easily conclude from Eqs. (5) and (13), taking account of Eq. (12), that the density \( \rho \) of the mixture is a constant. Since \( \rho \) has been proved to be constant, all of the coefficients in Eqs. (23) and (24) are constants. Now we shall seek a solution of the following form (Riedler and Schneider, 1983):

\[
\psi_{\beta}(r, \theta) = r f_{\beta}(\theta).
\]  

(52)

Inserting \( v_{\beta r} \) and \( v_{\beta \theta} \) from Eq. (51), with the aid of (52), into the \( r \)- and \( \theta \)-components of the momentum equations (23) and (24) and eliminating the pressure terms by cross-differentiating yields

\[
M_1 (f_{1r}'' + 2f_{1\theta}' + f_1) + M_2 (f_{2r}'' + 2f_{2\theta}' + f_2) - \alpha r^2 (f_{1r}'' - f_{2r}'' + f_1 - f_2) = 0,
\]

(53)

\[
M_3 (f_{1r}'' + 2f_{1\theta}' + f_1) + M_4 (f_{2r}'' + 2f_{2\theta}' + f_2) + \alpha r^2 (f_{1r}'' - f_{2r}'' + f_1 - f_2) = 0
\]

(54)

where primes denote differentiation with respect to \( \theta \).

Let us make the variables and material constants non-dimensional by the following substitutions:

\[
M_i = \frac{M_i}{\mu}, \quad \bar{\alpha} = \frac{\alpha r^2}{\mu}, \quad \bar{f}_{\beta}(\theta) = \frac{f_{\beta}(\theta)}{V}.
\]

(55)

Thus the non-dimensional governing equations become

\[
\bar{M}_1 (\bar{f}_{1r}'' + 2\bar{f}_{1\theta}' + \bar{f}_1) + \bar{M}_2 (\bar{f}_{2r}'' + 2\bar{f}_{2\theta}' + \bar{f}_2) - \bar{\alpha} (\bar{f}_{1r}'' - \bar{f}_{2r}'' + \bar{f}_1 - \bar{f}_2) = 0,
\]

(56)

\[
\bar{M}_3 (\bar{f}_{1r}'' + 2\bar{f}_{1\theta}' + \bar{f}_1) + \bar{M}_4 (\bar{f}_{2r}'' + 2\bar{f}_{2\theta}' + \bar{f}_2) + \bar{\alpha} (\bar{f}_{1r}'' - \bar{f}_{2r}'' + \bar{f}_1 - \bar{f}_2) = 0
\]

(57)

The boundary conditions for the dimensionless velocity fields are as follows:

\[
\bar{v}_{\beta r}(r, 0) = -1, \quad \bar{v}_{\beta r}(r, \theta_0) = 0,
\]

\[
\bar{v}_{\beta \theta}(r, 0) = V_\beta / V (V_\beta > 0, V > 0), \quad \bar{v}_{\beta \theta}(r, \theta_0) = 0.
\]

(58)

From Eqs. (51), (52) and (58), it follows that the boundary conditions for the function \( f_{\beta} \) are

\[
f_{\beta}(0) = -V_\beta / V, \quad f_{\beta}(\theta_0) = 0,
\]

\[
f_{\beta}'(0) = -1, \quad f_{\beta}'(\theta_0) = 0.
\]

(59)

From Eqs. (56) and (57), making simple algebraic calculations as in the previous section, we can obtain the following equations:
where \( \gamma \). Applying the boundary conditions (59) to Eqs. (63)-(65) separately, we find

\[
\eta_1 (f_1^{iv} + 2f_1'' + f_1) - \alpha (M_2 + M_4) (f_1'' - f_2'' + f_1 - f_2) = 0, \tag{60}
\]

\[
-\eta_1 (f_2^{iv} + 2f_2'' + f_2) - \alpha (M_1 + M_3) (f_1'' - f_2'' + f_1 - f_2) = 0. \tag{61}
\]

The sum of the above equations is

\[
f_1^{iv} - f_2^{iv} + (1 - \gamma_2) (f_1'' - f_2'') - \gamma_2 (f_1 - f_2) = 0 \tag{62}
\]

where \( \gamma_2 = -1 + \alpha \eta_2/\eta_1 \). The characteristic roots of Eq. (62) are \( \pm i \) and \( \pm \sqrt{\gamma_2} \). Hence, the general solution is

If \( \gamma_2 < 0 \),

\[
f_1 - f_2 = C_1 \cos \theta + C_2 \sin \theta + C_3 \cos(\sqrt{\gamma_2} \theta) + C_4 \sin(\sqrt{\gamma_2} \theta). \tag{63}
\]

If \( \gamma_2 = 0 \),

\[
f_1 - f_2 = D_1 \cos \theta + D_2 \sin \theta + D_3 \theta + D_4. \tag{64}
\]

If \( \gamma_2 > 0 \),

\[
f_1 - f_2 = E_1 \cos \theta + E_2 \sin \theta + E_3 \cosh(\sqrt{\gamma_2} \theta) + E_4 \sinh(\sqrt{\gamma_2} \theta). \tag{65}
\]

Applying the boundary conditions (59) to Eqs. (63)-(65) separately, we find

\[
C_1 = \frac{\sqrt{\gamma_2} (V_2 - V_1)}{C^*} \left[ \cos \theta_0 \cos(\sqrt{\gamma_2} \theta_0) + \sqrt{\gamma_2} \sin \theta_0 \sin(\sqrt{\gamma_2} \theta_0) - 1 \right],
\]

\[
C_2 = \frac{\sqrt{\gamma_2} (V_1 - V_2)}{C^*} \left[ \sqrt{\gamma_2} \cos \theta_0 \sin(\sqrt{\gamma_2} \theta_0) - \sin \theta_0 \cos(\sqrt{\gamma_2} \theta_0) \right],
\]

\[
C_3 = \frac{(V_1 - V_2)}{C^*} \left\{ \sqrt{\gamma_2} \left[ 1 - \cos \theta_0 \cos(\sqrt{\gamma_2} \theta_0) \right] - \sin \theta_0 \sin(\sqrt{\gamma_2} \theta_0) \right\},
\]

\[
C_4 = \frac{(V_1 - V_2)}{C^*} \left[ \sin \theta_0 \cos(\sqrt{\gamma_2} \theta_0) - \sqrt{\gamma_2} \cos \theta_0 \sin(\sqrt{\gamma_2} \theta_0) \right],
\]

\[
D_1 = D^* (\cos \theta_0 - 1), \quad D_2 = D^* \sin \theta_0, \quad D_3 = -D_2, \quad D_4 = D_1 + D^* \theta_0 \sin \theta_0,
\]

\[
E_1 = \frac{\sqrt{\gamma_2} (V_2 - V_1)}{E^*} \left[ 1 - \cos \theta_0 \cosh(\sqrt{\gamma_2} \theta_0) + \sqrt{\gamma_2} \sin \theta_0 \sinh(\sqrt{\gamma_2} \theta_0) \right].
\]
Having substituted Eqs. (63)-(65) into Eqs. (60) and (61), we integrate the resulting equations and obtain the following solutions for $f_1(\theta)$ and $f_2(\theta)$, respectively.

If $\gamma_2 < 0$,

$$f_1(\theta) = (C_5 + C_{10}) \cos \theta + (C_7 + C_{11}) \sin \theta - \frac{\alpha (M_2 + M_4)}{\eta_1 (\gamma_2 - 1)} \left[ C_3 \cos(\sqrt{\gamma_2} \theta) + C_4 \sin(\sqrt{\gamma_2} \theta) \right],$$

$$f_2(\theta) = (C_9 + C_{10}) \cos \theta + (C_{11} + C_{12}) \sin \theta + \frac{\alpha (M_1 + M_3)}{\eta_1 (\gamma_2 - 1)} \left[ C_3 \cos(\sqrt{\gamma_2} \theta) + C_4 \sin(\sqrt{\gamma_2} \theta) \right].$$  \hspace{1cm} (68)

If $\gamma_2 = 0$,

$$f_1(\theta) = (D_5 + D_{10}) \cos \theta + (D_7 + D_{11}) \sin \theta + \frac{\alpha (M_2 + M_4)}{\eta_1} (D_4 + D_3 \theta),$$

$$f_2(\theta) = (D_9 + D_{10}) \cos \theta + (D_{11} + D_{12}) \sin \theta - \frac{\alpha (M_1 + M_3)}{\eta_1} (D_4 + D_3 \theta).$$  \hspace{1cm} (69)

If $\gamma_2 > 0$,

$$f_1(\theta) = (E_5 + E_{10}) \cos \theta + (E_7 + E_{11}) \sin \theta + \frac{M_2 + M_4}{\eta_2} [E_3 \cosh(\sqrt{\gamma_2} \theta) + E_4 \sinh(\sqrt{\gamma_2} \theta)],$$

$$f_2(\theta) = (E_9 + E_{10}) \cos \theta + (E_{11} + E_{12}) \sin \theta - \frac{M_1 + M_3}{\eta_2} [E_3 \cosh(\sqrt{\gamma_2} \theta) + E_4 \sinh(\sqrt{\gamma_2} \theta)].$$  \hspace{1cm} (70)
With the help of Eq. (59), the constants of integration $C_5, ..., C_{12}$, $D_5, ..., D_{12}$ and $E_5, ..., E_{12}$ can be expressed as

$$C_5 = -\frac{V_1}{V} + \frac{\alpha (M_2 + M_4)}{\eta_1 (|\gamma_2| - 1)} C_3,$$

$$C_6 = \left\{ \left[ \alpha (M_2 + M_4) (V_1 - V_2) + \eta_1 V_1 (|\gamma_2| - 1) \right] \left[ \sin(2\theta_0) + 2\theta_0 \right] + 2 V_1 \eta_1 (|\gamma_2| - 1) \sin^2 \theta_0 \right\} \left/ \dot{C} \right.,$$

$$C_7 = 2 \alpha V (M_2 + M_4) \left\{ C_3 \left[ \left( \cos \theta_0 - \cos(\sqrt{\gamma_2} \theta_0) \right) - \sqrt{\gamma_2} \sin(\sqrt{\gamma_2} \theta_0) \right] \sin \theta_0 + \left( 1 - \cos \theta_0 \cos(\sqrt{\gamma_2} \theta_0) \right) \theta_0 \right] + C_4 \left[ \left( \sqrt{\gamma_2} \theta_0 \cos(\sqrt{\gamma_2} \theta_0) - \sin(\sqrt{\gamma_2} \theta_0) \right) \sin \theta_0 \right. \\
\left. + \left( \sqrt{\gamma_2} \theta_0 \cos \theta_0 \sin(\sqrt{\gamma_2} \theta_0) \right) \theta_0 \right] \left/ \dot{C} \right. \\
+ [\eta_1 V_1 (1 - |\gamma_2|) \sin(2\theta_0) + 2 \eta_1 \theta_0 (1 - |\gamma_2|) (V_1 + \theta_0 V)] \left/ \dot{C} \right.,$$

$$C_8 = \left\{ 2 \left[ \alpha (M_2 + M_4) (V_1 - V_2) + \eta_1 V_1 (|\gamma_2| - 1) \right] \sin^2 \theta_0 + \eta_1 V_1 (|\gamma_2| - 1) [2\theta_0 - \sin(2\theta_0)] \right\} \left/ \dot{C} \right.,$$

$$C_9 = C_5 - C_1, \quad C_{10} = C_6, \quad C_{11} = C_7 - C_2, \quad C_{12} = C_8,$$

$$D_5 = -\frac{V_1}{V} - \frac{\alpha (M_2 + M_4)}{\eta_1} D_4,$$

$$D_6 = \{ \left[ \alpha (M_2 + M_4) (V_2 - V_1) + \eta_1 V_1 \right] \left[ \sin(2\theta_0) + 2\theta_0 \right] + 2 \eta_1 V \sin^2 \theta_0 \} \left/ \dot{D} \right.,$$

$$D_7 = 2 \alpha V (M_2 + M_4) \left\{ D_3 \left( \cos \theta_0 - 1 \right) \theta_0^2 + D_4 \left[ (1 - \cos \theta_0 \sin(\theta_0 - \theta_0) \right) \right] \left/ \dot{D} \right. \\
- \eta_1 \left[ V_1 \left[ \sin(2\theta_0) + 2\theta_0 \right] + 2 V \theta_0^2 \right] \left/ \dot{D} \right.,$$

$$D_8 = \{ 2 \left[ \alpha (M_2 + M_4) (V_2 - V_1) + \eta_1 V_1 \right] \sin^2 \theta_0 + \eta_1 V [2\theta_0 - \sin(2\theta_0)] \} \left/ \dot{D} \right.,$$

$$D_9 = D_5 - D_1, \quad D_{10} = D_6, \quad D_{11} = D_7 - D_2, \quad D_{12} = D_8,$$

$$E_5 = -\frac{V_1}{V} - \frac{(M_2 + M_4)}{\eta_2} E_3,$$

$$E_6 = \{ \left[ (M_2 + M_4) (V_2 - V_1) + \eta_2 V_1 \right] \left[ \sin(2\theta_0) + 2\theta_0 \right] + 2 \eta_2 V \sin^2 \theta_0 \} \left/ \dot{E} \right.,$$
\[ E_7 = 2V(M_2 + M_4) \left\{ E_2 \left[ \cos(\sqrt{2}\theta_0) - \cos \theta_0 - \sqrt{2} \sin(\sqrt{2}\theta_0) \right] \cos \theta_0 \\
+ (\cos \theta_0 \cos(\sqrt{2}\theta_0) - 1) \theta_0 \right] + E_4 \left[ \sin(\sqrt{2}\theta_0) - \sqrt{2} \cos(\sqrt{2}\theta_0) \right] \sin \theta_0 \\
+ (\cos \theta_0 \sin(\sqrt{2}\theta_0) - \sqrt{2} \theta_0) \theta_0 \right\} \right\} / \dot{E} - [\eta_2 V_1 \sin(2\theta_0) + 2\eta_2 \theta_0 (V_1 + \theta_0 V)] / \dot{E}, \]

\[ E_8 = \left\{ 2 \left[ (M_2 + M_4) (V_2 - V_1) + \eta_2 V_1 \sin^2 \theta_0 + \eta_2 V [2\theta_0 - \sin(2\theta_0)] \right] \right\} / \dot{E}, \]

\[ E_9 = E_5 - E_1, \quad E_{10} = E_6, \quad E_{11} = E_7 - E_2, \quad E_{12} = E_8 \] (71)

where

\[ \dot{C} = \eta_1 V (\gamma_2 - 1) \cos(2\theta_0) + 2\theta_0^2 - 1, \quad \dot{D} = \eta_1 V [\cos(2\theta_0) + 2\theta_0^2 - 1], \]

\[ \dot{E} = \eta_2 V [\cos(2\theta_0) + 2\theta_0^2 - 1]. \] (72)

**Flow between two coaxial moving cylinders**

Finally, we study the fully developed flow of a binary mixture of incompressible Newtonian fluids between an inner cylinder of radius \( r_1 \) rotating at a constant rate of \( w_1 \) as well as translating at uniform velocity \( V \) and outer concentric cylinder of radius \( r_2 \) rotating at a constant rate of \( w_2 \), as sketched in Figure 3. The flow is driven by a combination of externally applied pressure gradients (\( \partial p_1 / \partial z, \partial p_2 / \partial z \)) and the motion of cylinders. Cylindrical coordinates \((r, \theta, z)\), with the \( z \)-axis coinciding with the common axis of the cylinders, are introduced.

We look for a solution, compatible with the mass balance equations (5), of the form

\[ \mathbf{v}_\beta = [0, v_{\beta\theta}(r), v_{\beta z}(r)], \quad \rho = \rho(r) \] (73)

where \( v_{\beta\theta} \) and \( v_{\beta z} \) denote the velocity components of the \( \beta \) th fluid in the directions of \( \theta \) and \( z \), respectively. Substituting velocity components and total density from Eq. (65) into equations of motion (23) and (24) gives

\[ \lambda \rho' = \frac{\partial p_1}{\partial r}, \]

*Figure 3. Schematic diagram of flow*

\[ M_1 (r^2 v'_{1\theta} + r v'_{2\theta} - v_{1\theta}) + M_2 (r^2 v'_{2\theta} + r v'_{2\theta} - v_{2\theta}) - \alpha r^2 (v_{1\theta} - v_{2\theta}) + r^2 (M_1 v'_{1\theta} + M_2 v'_{2\theta}) - r (M_1 v_{1\theta} + M_2 v_{2\theta}) = \frac{\partial p_1}{\partial \theta}, \] (75)

\[ M_1 v''_{1z} + M_2 v''_{2z} + \frac{1}{r} (M_1 v'_{1z} + M_2 v'_{2z}) + M_1 v'_{1z} + M_2 v'_{2z} - \alpha (v_{1z} - v_{2z}) = \frac{\partial p_1}{\partial z}, \] (76)
\[-\lambda \rho' = \frac{\partial p_2}{\partial r}, \quad (77)\]

\[M_3(r'v'_{10} + rv'_{10} - v_{10}) + M_4(r^2v''_{20} + rv'_{20} - v_{20}) + \alpha r^2(v_{10} - v_{20}) \]

\[+ r^2(M'_1v'_{10} + M'_2v'_{20}) - r(M'_1v_{10} + M'_2v_{20}) = r \frac{\partial p_2}{\partial \theta}, \quad (78)\]

\[M_3v''_{1z} + M_4v''_{2z} + \frac{1}{r}(M_3v'_{1z} + M_4v'_{2z}) + M'_1v'_{1z} + M'_2v'_{2z} + \alpha (v_{1z} - v_{2z}) = \frac{\partial p_2}{\partial z}. \quad (79)\]

The primes here indicate differentiation with respect to \(r\). Elimination of \(\partial \Lambda / \partial r\) between Eqs. (74) and (77), with the help of Eqs. (12), (16) and (20), leads to

\[(\rho - \rho_{10})(\rho_{20} - \rho) \frac{dp}{dr} \frac{d^2(\rho A)}{dp^2} = 0. \quad (80)\]

Here, in general, \(\rho \neq \rho_{10}, \rho \neq \rho_{20}\) and \(d^2(\rho A)/dp^2 \neq 0\), and hence we arrive at the conclusion that the total density \(\rho\) is a constant. Since \(\rho\) has been proved to be constant, the coefficients \(M_1, M_2, M_3, M_4, M'_1, M'_2\) appearing in equations of motion become constants. It also follows that the quantities \(p_1\) and \(p_2\) are not functions of \(r\). From Eqs. (75) and (78), it is clear that \(\partial p_1 / \partial \theta\) and \(\partial p_2 / \partial \theta\) are constants. Since \(p_1\) and \(p_2\) are periodic functions of \(\theta\), these constants must be equal to zero. Consequently, \(p_1\) and \(p_2\) can be at most functions of \(z\). Then Eqs. (76) and (79) imply that the pressure gradients are constants, i.e. \(\partial p_1 / \partial z = p_{10}\) and \(\partial p_2 / \partial z = p_{20}\). It is assumed that \(p_{10} = p_{20} = p_0\). In the light of these arguments, the equations of motion reduce to

\[M_1(r^2v''_{10} + rv'_{10} - v_{10}) + M_2(r^2v''_{20} + rv'_{20} - v_{20}) - \bar{\alpha} r^2(v_{10} - v_{20}) = 0, \quad (81)\]

\[M_1v''_{1z} + M_2v''_{2z} + \frac{1}{r}(M_1v'_{1z} + M_2v'_{2z}) - \bar{\alpha} (v_{1z} - v_{2z}) = p^*, \quad (82)\]

\[M_3(r^2v''_{10} + rv'_{10} - v_{10}) + M_4(r^2v''_{20} + rv'_{20} - v_{20}) + \bar{\alpha} r^2(v_{10} - v_{20}) = 0, \quad (83)\]

\[M_3v''_{1z} + M_4v''_{2z} + \frac{1}{r}(M_3v'_{1z} + M_4v'_{2z}) + \bar{\alpha} (v_{1z} - v_{2z}) = p^* \quad (84)\]

where

\[\bar{\alpha} = \frac{\alpha r^2}{\mu}, \quad p^* = \frac{p_0 r^2}{\mu V}. \quad (85)\]

From the above equations, after a little algebra as in the previous sections, we get

\[\eta_1 \left(v'_{1z} + \frac{1}{r}v'_{1z}\right) - \alpha (M_2 + M_4) (v_{1z} - v_{2z}) = (M_4 - M_2) p^*, \quad (86)\]

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\[-\eta_1 \left( v''_{2z} + \frac{1}{r} v'_z \right) - \alpha (M_1 + M_3) (v_{1z} - v_{2z}) = (M_3 - M_1) p^*, \quad (87)\]

\[\eta_1 (r^2 v''_{1\theta} + r v'_{1\theta} - v_1\theta) - \alpha r^2 (M_2 + M_4) (v_{1\theta} - v_{2\theta}) = 0, \quad (88)\]

\[-\eta_1 (r^2 v''_{2\theta} + r v'_{2\theta} - v_{2\theta}) - \alpha r^2 (M_1 + M_3) (v_{1\theta} - v_{2\theta}) = 0. \quad (89)\]

Adding Eqs. (86) and (87) gives
\[\eta_1 (v'_{1z} - v'_{2z}) + \frac{1}{r} \eta_1 (v'_{1z} - v'_{2z}) - \alpha \eta_2 (v_{1z} - v_{2z}) = \eta_4\]

where \(\eta_4 = -\eta_3 p^*\). Integration of this equation yields
\[v_{1z} - v_{2z} = C_1 I_0 (\gamma_1 r) + C_2 K_0 (\gamma_1 r) - \frac{\eta_4}{\alpha \eta_2}\]

where \(I_0\) and \(K_0\) are modified Bessel functions of order zero.

The no-slip boundary conditions of the problem are
\[v_{\beta \theta}(r_1/r_2) = \frac{r_1 w_1}{r_2 w_2}, \quad v_{\beta \theta}(1) = 1, \quad (92)\]

\[v_{\beta z}(r_1/r_2) = 1, \quad v_{\beta z}(1) = 0. \quad (93)\]

The boundary conditions on velocity given by Eq. (93) require

\[C_1 = \frac{\eta_4 [I_0(\gamma_1) - K_0(\gamma_1 r_1/r_2)]}{\alpha \eta_2 [I_0(\gamma_1 r_1/r_2) K_0(\gamma_1) - I_0(\gamma_1) K_0(\gamma_1 r_1/r_2)]}, \quad (94)\]

\[C_2 = \frac{\eta_4 [I_0(\gamma_1) - I_0(\gamma_1 r_1/r_2)]}{\alpha \eta_2 [K_0(\gamma_1 r_1/r_2) I_0(\gamma_1) - K_0(\gamma_1) I_0(\gamma_1 r_1/r_2)]}.\]

Substituting Eq. (91) into Eqs. (86) and (87), and integrating these differential equations, we have, respectively

\[v_{1z} = \frac{M_2 + M_4}{\eta_2} \left\{ C_1 [I_0(\gamma_1 r) - 1] + C_2 K_0(\gamma_1 r) \right\} + \frac{p^*}{2 \eta_2} r^2 + C_3 \ln r + C_4, \quad (95)\]

\[v_{2z} = -\frac{M_1 + M_3}{\eta_2} \left\{ C_1 [I_0(\gamma_1 r) - 1] + C_2 K_0(\gamma_1 r) \right\} + \frac{p^*}{2 \eta_2} r^2 + C_5 \ln r + C_6. \quad (96)\]

Here \(C_3, ..., C_6\) are constants of integration. Boundary conditions (93) allow us to express these constants as
The sum of Eqs. (88) and (89)

\[ r^2 (v'_{1\theta} - v'_{2\theta}) + r (v_{1\theta} - v_{2\theta}) - (1 + \gamma_1^2 r^2) (v_{1\theta} - v_{2\theta}) = 0. \]  

The solution of this equation is

\[ v_{1\theta} - v_{2\theta} = C_7 I_1 (\gamma_1 r) + C_8 K_1 (\gamma_1 r) \]  

where \( I_1 \) and \( K_1 \) are modified Bessel functions of order one. On application of conditions (92), we get

\[ v_{1\theta} - v_{2\theta} = 0. \]  

Substituting Eq. (100) into Eqs. (88) and (89), we obtain

\[ r^2 v_{\beta\theta}'' + r v_{\beta\theta}' - v_{\beta\theta} = 0. \]  

It is easy to see that \( \Omega_\beta \), under the boundary conditions (92), has the form

\[ v_{\beta\theta}(r) = \frac{[(r^2 - 1)w_1 + w_2] r_1^2 - r^2 w_2 r_2^2}{r (r_1^2 - r_2^2)w_2}. \]  

Discussion

In this paper some steady and slow flows of a mixture of two incompressible inert Newtonian fluids have been studied theoretically. Exact solutions have been obtained for the problems under consideration. We infer from these solutions that the presence of externally applied pressure gradients or the difference between boundary conditions for each fluid brings about the relative motion between the fluids, i.e. \( v_1 - v_2 \neq 0 \).

In order to make predictions based on the foregoing analysis, it is necessary to know all of the material functions in the constitutive equations. Determination of these functions for a mixture is much more difficult than that for a single continuum, owing to the large number of response functions appearing in the constitutive equations. On the other hand, a significant body of literature has grown up around the problem of determining these functions for a mixture is much more difficult than that for a single continuum, owing to the large number of response functions appearing in the constitutive equations. For example, employing results obtained from the kinetic theory of fluids, Sampaio and Williams (1977) were able to derive formulae for

\[ C_3 = \frac{M_2 + M_4}{\eta_2 \ln (r_1/r_2)} \{ C_1[I_0 (\gamma_1) - I_0 (\gamma_1 r_1/r_2)] + C_2[K_0 (\gamma_1) - K_0 (\gamma_1 r_1/r_2)] \} [1em] + \frac{p^* (r_2^2 - r_1^2) + 2 r_2^2 \eta_2}{2 r_2^2 \eta_2 \ln (r_1/r_2)}, \]

\[ C_4 = - \frac{M_2 + M_4}{\eta_2} \{ C_1[I_0 (\gamma_1) - 1] + C_2K_0 (\gamma_1) \} - \frac{p^*}{2 \eta_2}, \]

\[ C_5 = \frac{M_1 + M_3}{\eta_2 \ln (r_1/r_2)} \{ C_1[I_0 (\gamma_1 r_1/r_2) - I_0 (\gamma_1)] + C_2[K_0 (\gamma_1 r_1/r_2) - K_0 (\gamma_1)] \} + \frac{p^* (r_2^2 - r_1^2) + 2 r_2^2 \eta_2}{2 r_2^2 \eta_2 \ln (r_1/r_2)}, \]

\[ C_6 = \frac{M_1 + M_3}{\eta_2} \{ C_1[I_0 (\gamma_1) - 1] + C_2K_0 (\gamma_1) \} - \frac{p^*}{2 \eta_2}. \]
$\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ in terms of the viscosities of the unmixed fluids and the volume fractions in the case of $\lambda_3 = 0$. In this work, we benefit from the formulae suggested by Sampaio and Williams (1977) with the intention of assigning reasonable values to $M_1$, $M_2$, $M_3$ and $M_4$. To achieve this for a mixture composed of water and oil, at the outset we assume that the densities of unmixed fluids and the volume fractions are known. With the aid of Eqs. (3) and (10), knowledge of these quantities enables $\rho_1$, $\rho_2$ and $\rho = \rho_0$ to be calculated. Later, the viscosity coefficients can be determined by using the formulae proposed in the work of Sampaio and Williams (1977). For the purpose of simulations, the following values are given to the dimensionless parameters:

$$M_1 = 0.32, \quad M_2 = \bar{M}_3 = 0.22, \quad M_4 = 0.68.$$  

(103)

Figure 4. Velocity components in the $x$-direction for $Q = 1$, $\hat{\alpha} = 0.45$

Figure 5. Radial velocity components for $V_1/V = -0.1$, $V_2/V = -0.2$, $\hat{\alpha} = 1$, $\theta_0 = \pi/2$

Figure 6. Tangential velocity components for $V_1/V = -0.1$, $V_2/V = -0.2$, $\hat{\alpha} = 1$, $\theta_0 = \pi/2$

Figure 7. Streamline patterns for $V_1/V = -0.1$, $V_2/V = -0.2$, $\hat{\alpha} = 1$, $\theta_0 = \pi/2$

Figure 8. Axial velocity components for $\hat{\alpha} = 10$, $p^* = -5$
In Figures 4 to 9, the velocity distributions for constituents of binary mixture under consideration are plotted as a function of position for various values of the parameters, keeping the material constants fixed at the values given in Eq. (103). From these figures, we arrive at the conclusion that the particles of each constituent move independently with velocities \( \nu_1, \nu_2 \) at a given point in the mixture, but the velocity profiles of mixture components are generally similar to those of pure Newtonian fluids. For the solutions corresponding to pure Newtonian fluid we refer the reader to the books of Batchelor (1967) and Papanastasiou et al. (2000).

### References


