Flow of the Second-Order Rivlin-Ericksen Fluid Between Two Intersecting Planes

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Received: 28/7/1995

Abstract: In this paper, we worked on the non-radial flow of the Second-order Rivlin-Ericksen fluid between two intersecting planes. The role of the Reynolds number and the angle of the planes are carefully delineated. We find that there is an interesting structural change in the streamline patterns with variations of Reynolds number and angle of the planes.

Key Words: Intersecting Planes, Non-radial Flow, Streamline Patterns, the Second-order Rivlin-Ericksen Fluid.

İkinci Mertebeden Rivlin-Ericksen Akışının Kesişen iki düzlem arasındaki Akımı


Introduction

Jeffery (1915) and Hamel (1917), worked on the Navier-Stokes fluid between two intersecting planes. In their study, flow is assumed to be purely radial. If the inertial terms are ignored, it is possible to obtain an exact solution to the creeping flow equation (Birkhoff and Zarantonello, 1957). In the case of most non-Newtonian fluids a purely radial flow is not possible if inertial terms are to be retained in the equations of motion (Moffat and Duffy, 1980). Hull (1981), investigated the flow of a general linear viscoelastic fluid between intersecting planes and he showed that radial flow is possible, if only the planes make an angle of π/2. Straus (1974), studied the flow of Maxwell fluid between intersecting planes. He assumed that flow is non-radial. In his subsequent study, he worked on the stability of the flow between intersecting planes (Strauss, 1975). Bhatragar et al., (1993), worked on the Oldroyd-B flow between intersecting planes. Recently, Öztürk et al., (1995), studied a special type of Oldroyd-B flow between intersecting planes. Here, we study the flow of the second-order Rivlin-Ericksen fluid between intersecting planes. We have met no non-dimensional parameter depending the material constants of the fluid; the only non-dimensional parameter is the Reynolds number.

Constitutive Equation

The Cauchy stress tensor \( T \) for the second-order Rivlin-Ericksen fluid is related to the fluid motion in the following manner (Rivlin and Ericksen, 1955).

\[
T = -pI + S
\]

\[
S = \mu (A_1 + \alpha_1 A_1^T + \alpha_2 A_2)
\]

where \(-pI\) is the constitutively indeterminate part of the stress due to constraint of incompressibility, \( S \) is the extra stress tensor, \( \mu \) is the viscosity, \( \alpha_1 \) and \( \alpha_2 \) are material constants which are referred to as the cross viscosity and elastic coefficient, respectively. The kinematical tensors \( A_1 \) and \( A_2 \), which are Rivlin-Ericksen tensors, are defined through

\[
A_1 = 2D = L + L^T
\]

\[
A_2 = \frac{d}{dt} A_1 + L A_1 + A_1 L^T
\]
\[ L = \nabla \mathbf{v}, \quad L_{ij} = \nu_{ij} \]  

where \( \mathbf{D} \) is the rate of deformation tensor, \( \mathbf{v} \) is the velocity vector, \( d/dt \) denotes the material time derivative which is defined as follows:

\[ \frac{d}{dt} \left[ \mathbf{v} \right] = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \]  

and comma in the suffix denotes covariant differentiation. \( A_k \) is now in the form

\[ A_2 = \nabla \mathbf{a} + [\nabla \mathbf{a}]^T \cdot [L_{ij} L_{ij}] + L A_1 + A_1 L^T \]  

Using equation (3) and (5) in equation (7) we have

\[ A_2 = \nabla \mathbf{a} + [\nabla \mathbf{a}]^T + \frac{2}{r} [\nabla \mathbf{v}] [\nabla \mathbf{v}]^T \]  

where \( \mathbf{a} \) is the acceleration vector. When \( \alpha_1 = \alpha_2 = 0 \), the model (2) reduces to the classical linearly viscous model, while \( \alpha_2 = 0 \) it reduces to the Reiner model. For the physical model of the problem (See Figure 1).

We shall assume a velocity field in a cylindrical polar coordinate system of the form

\[ \mathbf{v}(r, \theta, z) = u(r, \theta)e_r + v(r, \theta)e_\theta \]  

Thus, we take into account the fact that the flow is not radial, though planar. It follows from the equations (1) through (9) that the constitutive equations for the fluid under consideration take the forms

\[ \theta = \pm \alpha \]  

\[ \theta = 0 \]  

\[ \theta = -\alpha \]  

Balance Equations

The conservation equation of mass is now in the form

\[ \frac{d}{dr} (ru) + \frac{\partial v}{\partial \theta} = 0 \]  

and the equation of motion takes the forms of
\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}}{\partial x} & = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \\
\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}}{\partial x} & = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \rho \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \right) \\
\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{w}}{\partial x} & = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \rho \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} \right)
\end{align*}
\]

(14)

We shall find it convenient to use the non-dimensional field of equations (10)-(15). To this end, let us introduce the flux \(q\) through

\[
q = \int u \, r \, d \theta
\]

(16)

Notice that \(q\) has dimensions of \(L^2 T^{-1}\), and we define dimensionless fields as follows

\[
\bar{u} = \sqrt{\frac{\nu}{\nu_q}} \, u, \quad \bar{v} = \sqrt{\frac{\nu}{\nu_q}} \, v, \quad \bar{r} = \sqrt{\frac{\nu}{\nu_2}} \, r
\]

\[
\bar{T}_r = \frac{v_2}{v_\mu} \, T_r, \quad \bar{p} = \frac{v_2}{v_\mu} \, p
\]

where \(v = \nu/\nu, v_1 = \alpha_1 \nu, v_2 = \alpha_2 \nu\)

(17)

It follows from (10)-(15) and using (17)-(18) that (on dropping the bars for convenience)

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} & = \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} + \frac{\partial \bar{u}}{\partial \theta} \frac{\partial \bar{u}}{\partial \theta} + \frac{1}{\rho} \left( \frac{\partial \rho}{\partial x} \frac{\partial \bar{u}}{\partial x} + \frac{\partial \rho}{\partial y} \frac{\partial \bar{u}}{\partial y} + \frac{\partial \rho}{\partial z} \frac{\partial \bar{u}}{\partial z} \right) \\
\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \bar{w} \frac{\partial \bar{v}}{\partial z} & = \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{v}}{\partial z^2} + \frac{\partial \bar{v}}{\partial \theta} \frac{\partial \bar{v}}{\partial \theta} + \frac{1}{\rho} \left( \frac{\partial \rho}{\partial x} \frac{\partial \bar{v}}{\partial x} + \frac{\partial \rho}{\partial y} \frac{\partial \bar{v}}{\partial y} + \frac{\partial \rho}{\partial z} \frac{\partial \bar{v}}{\partial z} \right) \\
\frac{\partial \bar{w}}{\partial t} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{v} \frac{\partial \bar{w}}{\partial y} + \bar{w} \frac{\partial \bar{w}}{\partial z} & = \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial z^2} + \frac{\partial \bar{w}}{\partial \theta} \frac{\partial \bar{w}}{\partial \theta} + \frac{1}{\rho} \left( \frac{\partial \rho}{\partial x} \frac{\partial \bar{w}}{\partial x} + \frac{\partial \rho}{\partial y} \frac{\partial \bar{w}}{\partial y} + \frac{\partial \rho}{\partial z} \frac{\partial \bar{w}}{\partial z} \right)
\end{align*}
\]

(19)

where the Reynolds number \(Re\) and material parameter \(\sigma_1\) are defined through

\[
Re = \frac{\bar{u}}{v}, \quad \sigma_1 = \frac{\alpha_1 \bar{v}}{\nu_2}
\]

(20)

By cross-differentiating (23) and (24) we eliminate the pressure and obtain the following governing equation

\[
\frac{\partial^2 \bar{T}_r}{\partial \theta^2} + \frac{2}{\bar{r}} \frac{\partial \bar{T}_r}{\partial \bar{r}} = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial x} \frac{\partial \bar{T}_r}{\partial x} + \frac{\partial \rho}{\partial y} \frac{\partial \bar{T}_r}{\partial y} + \frac{\partial \rho}{\partial z} \frac{\partial \bar{T}_r}{\partial z} \right) - \bar{u} \frac{\partial^2 \bar{T}_r}{\partial \bar{r}^2} - \frac{1}{\rho} \left( \frac{\partial \rho}{\partial x} \frac{\partial \bar{T}_r}{\partial x} + \frac{\partial \rho}{\partial y} \frac{\partial \bar{T}_r}{\partial y} + \frac{\partial \rho}{\partial z} \frac{\partial \bar{T}_r}{\partial z} \right)
\]

(21)

Next, we introduce the streamline function \(\psi(r, \theta)\) through

\[
u = \frac{1}{\rho} \frac{\partial \psi}{\partial \theta}, \quad r = \frac{\partial \psi}{\partial \bar{r}}
\]

(22)

(23)

(24)

(25)

(26)

(27)
and we shall also express \( \psi, T_r, T_\theta \) and \( T_{r\theta} \) as a power series of the form (Strauss, 1974).

\[
\psi(r,\theta) = \sum_{n=0}^{\infty} \frac{a_n(\theta)}{r^n}, \quad T_r = \sum_{n=0}^{\infty} \frac{a_n(\theta)}{r^n}, \\
T_\theta = \sum_{n=0}^{\infty} \frac{a_n(\theta)}{r^n}, \quad T_{r\theta} = \sum_{n=0}^{\infty} \frac{a_n(\theta)}{r^n}
\]

(28)

The series solution (28) that is sought is not akin to a perturbation, for at values of \( r<1 \), the higher-order terms in the series can make a much more significant contribution to the solution than the lower-order terms. Since the expansion (28) is singular at \( r=0 \), and this is so in the Navier-Stokes fluid as well, the results are expected to hold only in the converging and diverging nozzles rather than in between intersecting planes as we would need to have in place a source or a sink at \( r=0 \). Since we have used the radial coordinate dimensionless by using the length scale \( \sqrt{\frac{V}{\nu \lambda} } \), that is \( \bar{r} = \sqrt{\frac{V}{\nu \lambda} } r \), the domain of the validity of the solution in terms of the dimensional radial coordinate \( r \) depends on the magnitude of \( \sqrt{\frac{V}{\nu \lambda} } \).

So we can say that, if the value \( \sqrt{\frac{V}{\nu \lambda} } \) is appropriately large, the results could be valid for values of \( r \) smaller than unity. On the other hand for \( r<<1 \), the domain of the solution depends on the function \( \psi \) that is, if \( \psi \) are not identically zero for large \( n \), the solution cannot be trusted.

Boundary conditions of the problem are follows:

\[
u = 0, \quad v = 0, \quad \text{when } \theta = \pm \alpha.
\]

(29)

which by virtue of (27) and (28) implies that

\[
\psi_n (\pm \alpha) = 0, \quad n = 0, 1, 2, \ldots
\]

(30)

We need two additional boundary conditions at the zeroth-order. If we are in a position to solve for \( \psi_n \), that is, we must know the value of \( \psi_n (\pm \alpha) \). We use the normalization condition and assuming the flow is symmetric about \( \theta=0 \), then we have

\[
\psi_0 (\alpha) = -1/2, \quad \psi_0 (-\alpha) = 1/2
\]

(31)

which determines the mass flux. So, our solution is for symmetric state, of course, for a certain mass flux there are solutions that have no symmetry.

We now turn our attention to the series solution (28).

On using (27) and the representations (28) for \( T_r, T_\theta \) and \( T_{r\theta} \) in (19)-(21) and entering all these into (26), a very long and tedious calculation yields the equations a various orders of \( r^n \). Here, we carry out our calculations up to order \( n=4 \). We do not prove convergence of the series but such a formal analysis is in keeping with perturbation analysis in non-Newtonian fluid mechanics.

Solutions

Zeroth-order solution

The zeroth-order problem is governed by

\[
\psi_0'' + 2 \text{Re} \psi_0 \psi_0' + 4 \psi_0 = 0,
\]

(32)

\[
\psi_0 (\pm \alpha) = 0, \quad \psi_0 (\alpha) = -1/2, \quad \psi_0 (-\alpha) = 1/2.
\]

(33)

Differential equation of zeroth-order problem is non-linear and the only parameter is Reynolds number. Equation (32) subject to (33) is solved numerically.

First-order solution

Differential equation governing \( \psi_1 (\theta) \) is given as follows

\[
\psi_1'' + [10 + 3 \text{Re} \psi_0] \psi_1 + 2 \text{Re} \psi_0 \psi_1 + [9 + \text{Re} [2 \psi_0 \psi_0''] \psi_1 = 0.
\]

(34)

This equation subject to

\[
\psi_1 (\pm \alpha) = 0, \quad \psi_1 (\pm \alpha) = 0.
\]

(35)

We notice that (34) is a linear homogeneous ordinary differential equation subject to (35) and solution is

\[
\psi_1 (\theta) = 0.
\]

(36)

Second-order solution

Differential equation governing \( \psi_2 (\theta) \) is given as follows

\[
\psi_2'' + [20 + 4 \text{Re} \psi_0] \psi_2 + 2 \text{Re} \psi_0 \psi_2 + [64 - \text{Re} [2 \psi_0 \psi_0''] \psi_2 = 4 (4 \psi_0 + \psi_0'') \psi_0.
\]

(37)

The right hand side of (37) represents non-Newtonian character of the fluid. In the case of slow
motion there is no non-Newtonian effect. This equation subject to

\[ \psi_2(\pm \alpha) = 0, \quad \psi_3(\pm \alpha) = 0. \]  

Equation (37) subject to (38) is solved numerically.

Third-order solution

Differential equation governing \( \psi_3(\theta) \) is given as follows

\[ \psi_{3}^{\prime\prime} + [3+5 \text{Re}\psi_0] \psi_3' + 2 \text{Re}\psi_0 \psi_3 + 
   + [225 - 3 \text{Re}\psi_0] \psi_3 = 0 \]

This equation subject to

\[ \psi_3(\pm \alpha) = 0, \quad \psi_3(\pm 2\pi) = 0. \]  

This is a linear homogeneous ordinary differential equation and it gives a zero solution under the boundary conditions.

\[ \psi_3(\theta) = 0. \]  

Fourth-order solution

Differential equation governing \( \psi_4(\theta) \) is given as follows

\[ \psi_{4}^{\prime\prime\prime} + [52 + 6 \text{Re}\psi_0] \psi_4' + 2 \text{Re}\psi_0 \psi_4 + [576 - 4 \text{Re}] \psi_4 + 
   - 96 \psi_0 \psi_4' + \text{Re} \{ -4 \psi_2 \psi_2' + 8 \psi_2 + 2 \psi_2 \} + 16 \psi_0 \psi_2' + 
   + 120 \psi_0 \psi_2' + 8 \psi_0 \psi_2 + 384 \psi_0 \psi_2 + 6 \psi_0 \psi_2' + 4 \psi_0 \psi_2 - 2 \psi_0 \psi_2:
\]

The terms on the right hand side of (42) having no Re factor represents the non-Newtonian character of the fluid. In the case of slow motion (Re=0) the effect of the non-Newtonian character of the fluid is nothing. Rivlin-Ericksen fluid gives no solution for slow motion. This equation subject to

\[ \psi_4(\pm \alpha) = 0, \quad \psi_4(\pm 2\pi) = 0. \]  

Equation (42) subject to (43) is solved numerically.

Numerical Solutions

Several numerical methods can be used to solve above differential equations. One convenient and accurate method which we will use here is the so-called shooting method for equation (32) subject to (33). Equation (32) and boundary conditions (33) are reduced to four first-order differential equations. For given values of \( \text{Re} \) and \( \alpha \), the conditions \( \psi_4(-\alpha) \) and \( \psi_0(-\alpha) \) are roughly estimated and differential equation is processed by using the fourth-order Runge-Kutta procedure. The mathematical problem is to find the correct values of \( \psi_4(-\alpha) \) and \( \psi_0(-\alpha) \) which yield the known values of \( \psi_0(\alpha) \) and \( \psi_0(\alpha) \) at terminal point. Since for Re=0 the analytic solution provides exact initial values for \( \psi_0(-\alpha) \) and \( \psi_0(-\alpha) \) then successive numerical solution can be generated as Re is increased.

The systematic way used here for finding values of the missing initial conditions is equivalent to a modified Newton's method, for finding the roots of equations in several variables.

Numerical calculations have been performed by using computer with the help of Turbo-Pascal 7.0 programs and we worked with real numbers of extended type. The chosen “h” (Step-length) allows a reasonably small truncation error without causing roundoff problems. The accuracy of missing initial conditions at \( \psi_0(-\alpha) \) and \( \psi_0(-\alpha) \) which yield the known values at the terminal point is \( 10^{-6} \) at least. The results have been summarized in Table 1.

Linear differential equations (37) and (42) together with the associated boundary conditions (38) and (43) are also two point boundary value problems which have been solved numerically with help of finite difference method. The range of \( 2\pi \) has been divided120 finite intervals, at a distance “h” apart. Approximation of the solution at nodal points has been obtained by solving the resulting linear algebraic equations with help of computer for various values of \( \text{Re} \) and \( \alpha \).

<table>
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<tr>
<th>( \text{Re} )</th>
<th>( \alpha )</th>
<th>( h )</th>
<th>( \psi_0(-\alpha) )</th>
<th>( \psi_0(-\alpha) )</th>
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</tr>
<tr>
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</table>
Results and Discussions

Our main purpose is to delineate the effect on the flow of the Reynolds number and the angle of intersection. For this reason we considered three different angle of $\alpha=15, 30$ and $60$ degree. Strauss (1974) and Bhatnagar et al. (1993), worked on the same problem for the flow of a Maxwell and oldroyd-b fluid, respectively. In general, our results are similar to that of Strauss (1974) and Bhatnagar et al. (1993). The curves in the figures are obtained for the constant stream function

$$\psi = \psi_0(\theta) + \frac{\psi_1(\theta)}{r^2} + \frac{\psi_2(\theta)}{r^4}$$

It is easy to infer from the structure of the streamlines that the flow in the distant region from the corner and close to the corner are quite different.

For instance, in the flow depicted in Figure 2-8, fluid from infinity mostly intends towards the apex of the wedge tangential to the bounding planes, with a narrow region of purely radial flow from infinity adjacent to the central plane ($\theta=0$) implying a two-cell structure. On the other hand, in Figure 9-10 we see that the flow in the distant region from the apex is strictly divided into four distinct domains, indicating presence of the four cells structure of streamlines in the region close to the apex.

In the cases of acute angle, the structure of the flow is significantly different from that of obtuse angle. For instance, as is seen in Figure 2-7 for the acute angles and for all values of the Reynolds number flow has a two-cell structure, but for the obtuse angle (See Figure 8-10) has four-cell structure for the large Reynolds number and two-cell structure for the
small values of the Reynolds number. So, for this kind of geometry the angle of intersection and the Reynolds number are both significantly important. The shape of the circulating cells are different at the large Reynolds number, they get skewed towards the wall for all angle of intersection. Increasing the Reynolds number produces a change in the streamline pattern. That is, the size of circulating cells being smaller, also especially in the case of $\alpha=120^\circ$ cells are skewed differently. Finally, we see that increasing Reynolds number results in pushing the cells towards the boundaries and the number of cell families being determined by the specific value of the Reynolds number and the angle of intersection of the planes. As is indicated above, we have met the effect of non-Newtonian character of the fluid in equation (37) and (42), but there is no effect of the material parameter $\sigma$, (related to the cross viscosity).

Symbols

$D$ Deformation-rate tensor
$A_1$ Rivlin-Ericksen tensor of rank one
$A_2$ Rivlin-Ericksen tensor of rank two
$I$ Unit tensor
$S$ Extra stress tensor
$T$ Cauchy stress tensor
$a$ Acceleration vector
$h$ Step-length
$p$ Pressure
$q$ Flux
$v$ Velocity vector
$\alpha_1$ Cross viscosity coefficient
$\alpha_2$ Elastic coefficient
$\rho$ Density
$\mu$ Viscosity
$\psi$ Streamline function

References


Öztürk, Y. et al., "Flow of Viscoplastic Fluid between Intersecting Planes", will appear in the proceedings of the third symposium of the University of Balikesir, Turkey, 1995.

